TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 350, Number 5, May 1998, Pages 1713–1790
S 0002-9947(98)01763-2

BOUNDARY SLOPES OF PUNCTURED TORI IN 3-MANIFOLDS

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Abstract. Let M be an irreducible 3-manifold with a torus boundary component T, and suppose that r, s are the boundary slopes on T of essential punctured tori in M, with their boundaries on T. We show that the intersection number $\Delta(r, s)$ of r and s is at most 8. Moreover, apart from exactly four explicit manifolds M, which contain pairs of essential punctured tori realizing $\Delta(r, s) = 8, 8, 7$ and 6 respectively, we have $\Delta(r, s) \leq 5$. It follows immediately that if M is atoroidal, while the manifolds $M(r), M(s)$ obtained by r- and s-Dehn filling on M are toroidal, then $\Delta(r, s) \leq 8$, and $\Delta(r, s) \leq 5$ unless M is one of the four examples mentioned above.

Let $\mathcal{H}_0$ be the class of 3-manifolds M such that M is irreducible, atoroidal, and not a Seifert fibre space. By considering spheres, disks and annuli in addition to tori, we prove the following. Suppose that $M \in \mathcal{H}_0$, where ∂M has a torus component T, and $\partial M - T \neq \varnothing$. Let r, s be slopes on T such that $M(r), M(s) \notin \mathcal{H}_0$. Then $\Delta(r, s) \leq 5$. The exterior of the Whitehead sister link shows that this bound is best possible.

1. Introduction

Let M be a 3-manifold and T a torus component of ∂M. (Throughout, all 3-manifolds will always be assumed to be compact, connected, and orientable.) With no real loss of generality, we shall assume that M is irreducible. Recall that the *slope* of an essential unoriented simple closed curve on T is its isotopy class, and that if r and s are two slopes on T then $\Delta(r, s)$ denotes their minimal geometric intersection number.

Let $(F, \partial F) \subset (M, \partial M)$ be an essential surface with $\partial F \cap T \neq \emptyset$. Then all the components of $\partial F \cap T$ have the same slope on T, the *boundary slope* of F on T. Consideration of such surfaces naturally arises in the context of Dehn filling. More precisely, if r is a slope on T, let $M(r)$ be the manifold obtained from M by r-*Dehn filling*, that is, by attaching a solid torus V to M along T so that r bounds a disk in V. If $M(r)$ contains an essential surface S, then either S can be moved off V into M, or there is an essential surface F in M with boundary slope r on T such that $\widehat{F}$, the surface obtained from F by capping off the components of $\partial F \cap T$ with disks, is homeomorphic to S. In this paper we shall be mainly concerned with the case where $\widehat{F}$ is a torus; let $\mathcal{T}(M, T)$ denote the set of boundary slopes on T of such punctured tori F in M.

Interesting examples are provided by some of the manifolds obtained by Dehn surgery on one component of the Whitehead link. Let W be the exterior of the Whitehead link and let T_0 be a boundary component of W. Choosing a standard

Received by the editors March 27, 1995 and, in revised form, February 26, 1996.
1991 *Mathematics Subject Classification.* Primary 57M25; Secondary 57M50.

meridian-longitude basis μ, λ for $H_1(T_0)$, we may parametrize the slopes on T_0 by $\mathbb{Q} \cup \{1/0\}$ in the usual way, by associating r with p/q if $[r] = p\mu + q\lambda \in H_1(T_0)$, and write $W(r) = W(p/q)$.

Then [Ho] for $M = W(2)$, $W(-5/2)$, $W(-5)$, and $W(1)$, $\operatorname{int} M$ is hyperbolic (so M is irreducible and contains no essential torus), whilst there exist slopes r, s on ∂M with $\Delta(r,s) = 6, 7, 8$, and 8 respectively such that $M(r)$ and $M(s)$ each contains an essential torus. Hence $r, s \in \mathcal{T}(M, \partial M)$. Our main result asserts that these examples are extremal.

Theorem 1.1. *Let M be an irreducible 3-manifold and T a torus component of ∂M. If $r, s \in \mathcal{T}(M,T)$ then either*

(1) $\Delta(r,s) \leq 5$*; or*
(2) $\Delta(r,s) = 6$ *and M is homeomorphic to $W(2)$; or*
(3) $\Delta(r,s) = 7$ *and M is homeomorphic to $W(-5/2)$; or*
(4) $\Delta(r,s) = 8$ *and M is homeomorphic to either $W(1)$ or $W(-5)$.*

We remark that $W(1)$ is homeomorphic to the exterior of the figure eight knot, and $\operatorname{vol} W(1) = \operatorname{vol} W(-5)$, $\operatorname{vol} W(-5/2)$, $\operatorname{vol} W(2)$ are respectively the lowest, second lowest, and third lowest known volumes of hyperbolic 3-manifolds with a single cusp. Also, in each case there are exactly two slopes r, s with the stated property.

Theorem 1.1 has an obvious corollary about the creation of essential tori by Dehn filling. It is natural to consider also how other surfaces of non-negative euler characteristic might arise, that is, the cases where $\widehat{F}$ is a sphere, disk or annulus. For instance, let us consider tori and spheres. As an example here, let M be the exterior of the trefoil knot. Then (parametrizing slopes on ∂M by meridian-longitude coordinates) $M(0)$ is a torus bundle over the circle, and $M(6)$ is the connected sum of the lens spaces $L(2,1)$ and $L(3,1)$. We then have the following result. Let $\mathcal{A}$ denote the class of irreducible 3-manifolds that contain no essential tori.

Theorem 1.2. *Suppose that $M \in \mathcal{A}$ and that ∂M has a torus component T. Let r, s be slopes on T such that $M(r), M(s) \notin \mathcal{A}$. Then either one of conclusions (1)–(4) of Theorem 1.1 holds, or $\Delta(r,s) = 6$ and M is homeomorphic to the exterior of the trefoil knot.*

Using Thurston's uniformization theorem for Haken manifolds [T], we obtain a bound on the degeneration of hyperbolic structures under Dehn filling, at least in the case where the resulting manifold has non-empty boundary. To best state the result, let $\mathcal{H}$ denote the class of 3-manifolds M such that $\operatorname{int} M$ has a complete hyperbolic structure, and let $\mathcal{H}_0$ be the set of elements of $\mathcal{H}$ that are not Seifert fibre spaces. Thus $\mathcal{H}_0 = \mathcal{H} - \{S^1 \times D^2, T^2 \times I\}$. Thurston has shown [T] that if M is a 3-manifold with non-empty boundary, then $M \in \mathcal{H}_0$ if and only if $M \in \mathcal{A}$ and is not a Seifert fibre space. By considering annuli and disks in addition to spheres and tori, we obtain the following theorem.

Theorem 1.3. *Suppose that $M \in \mathcal{H}_0$, where ∂M has a torus component T and at least one other component. Let r, s be slopes on T such that $M(r), M(s) \notin \mathcal{H}_0$. Then $\Delta(r,s) \leq 5$.*

Theorem 1.3 applies in particular to hyperbolic links in S^3 with at least two components. In fact, there is an example of a two-component link in S^3 which shows that the bound of 5 in Theorem 1.3 (and Theorems 1.1 and 1.2) cannot be improved. This is the *Whitehead sister* link illustrated in [BFLW, Figure 3]; one component is unknotted and the other is a trefoil. Its exterior M has a hyperbolic structure with the same volume as the exterior of the Whitehead link. For each of the two boundary components T of M, there are slopes r, s on T with $\Delta(r, s) = 5$ such that $M(r)$ and $M(s)$ contain essential tori [Ho].

In the case of Dehn filling on a hyperbolic manifold M with a single boundary component, one conjectures of course that if $M \in \mathcal{H}_0$ and $M(r), M(s) \notin \mathcal{H}_0$, then $\Delta(r, s) \leq 8$. In general, our topological methods give no information on this beyond Theorem 1.2. (Recall that it is a special case of Thurston's geometrization conjecture that if $M \in \mathcal{A}$ and $\pi_1(M)$ is infinite, then either $M \in \mathcal{H}$ or M belongs to a certain small class of Seifert fibre spaces.) Using Thurston's uniformization theorem for closed Haken manifolds, however, we get a result in the following special case.

Theorem 1.4. *Suppose that $M \in \mathcal{H}$ and that ∂M is a torus. Suppose that* $\operatorname{int} M$ *contains a closed incompressible surface S such that there is no incompressible annulus in M joining S to ∂M. Let r, s be slopes on ∂M such that $M(r), M(s) \notin \mathcal{H}$. Then $\Delta(r, s) \leq 5$.*

We now give a brief description of the organization of the paper.

As in [GLi], which considers boundary slopes of punctured spheres, a special role in our arguments is played by those pairs (M, T) that are cabled (see Section 2 for the definition). More specifically, we prove Theorem 1.1 in two parts, in the following propositions.

Proposition 1.5. *If (M, T) is not cabled and $r, s \in \mathcal{T}(M, T)$, then the conclusion of Theorem 1.1 holds.*

Proposition 1.6. *If (M, T) is cabled and $r, s \in \mathcal{T}(M, T)$, then $\Delta(r, s) \leq 4$.*

We remark that the bound of 4 in Proposition 1.6 is also sharp (see Section 13).

The main part, Proposition 1.5, is proved by analysing in detail the possible patterns of intersection of two punctured tori in M.

In Section 2 we prove some general lemmas about intersections of essential surfaces, whilst Section 3 contains the results we need about graphs in surfaces. The "generic" case of Proposition 1.5 is done in Section 4; the argument here is based on [GLi]. The completion of the proof of Proposition 1.5 occupies Sections 5–11. After some preparatory lemmas in Section 5, Sections 6–10 treat the various cases that remain after the results of Section 4. The conclusion is that if (M, T) is not cabled and $(F_\alpha, \partial F_\alpha) \subset (M, T)$ is an essential punctured torus with boundary slope r_α, $\alpha = 1, 2$, such that $\Delta = \Delta(r_1, r_2) \geq 6$, then (after isotoping the surfaces so as to minimize their intersection) the only possibilities for $(F_1, F_2; F_1 \cap F_2)$ are: one with $\Delta = 6$, an infinite family with $\Delta = 6$, one with $\Delta = 7$, and two with $\Delta = 8$. In Section 11 we show that for the infinite family with $\Delta = 6$, one of the surfaces F_α must actually be inessential, whilst in each of the other four cases the triple $(M; F_1, F_2)$ is uniquely determined. These must therefore correspond to the four examples listed in Theorem 1.1.

In Section 12 we consider other pairs of essential surfaces F_α in M such that $\widehat{F}_\alpha$ has non-negative euler characteristic, $\alpha = 1, 2$, and show, modulo cabling (or, in some cases, modulo the existence of essential annuli in M), that $\Delta(r_1, r_2) \leq 5$. In general this bound is not best possible, and in some cases stronger results are known, but as it is sufficient for Theorems 1.2, 1.3, and 1.4 we do not discuss the matter further. (For some recent results on these questions see [Wu1], [GLu], [BZ1], [BZ2], [H] and [HM].)

In Section 13 we consider the case where (M, T) is cabled, proving Proposition 1.6 and analogs for other surfaces of non-negative euler characteristic. The argument here makes use of the classification of essential planar surfaces in cable spaces given in [GLi]. We remark that for many applications, for example to Theorems 1.2, 1.3, and 1.4, consideration of cabled manifolds is not really necessary, as such a manifold either contains an essential torus or is a very special Seifert fibre space. However, expressed in terms of boundary slopes, the ultimate results apply (in most cases) equally well to cabled manifolds, so we carry out an analysis of the cabled case that is sufficient to give, for tori, Theorem 1.1, and for other surfaces of non-negative euler characteristic, the results stated in Section 14, Theorems 14.2, 14.3 and 14.5.

Finally, also in Section 14, we prove Theorems 1.2, 1.3, and 1.4.

I should like to thank Craig Hodgson for his help in telling me about the examples $W(-5)$, $W(-5/2)$, $W(2)$, and the Whitehead sister. The information given above about them and their toral boundary slopes was obtained by him using Jeffrey Weeks' Dehn surgery computer program "`snappea`." The manifold $W(-5)$ is also discussed at length in Weeks' thesis [We].

2. Intersections of essential surfaces

By a *surface* we shall mean a compact, connected, orientable 2-manifold, and we shall say that a surface in a 3-manifold is *essential* if it is properly embedded and either (i) incompressible, not parallel to a subsurface of the boundary of the 3-manifold, and not a 2-sphere, or (ii) a 2-sphere that does not bound a 3-ball.

Now let M be an irreducible 3-manifold (one that does not contain an essential 2-sphere), and let T be a torus component of ∂M. We assume that T is incompressible. (If not, M is a solid torus.) Let F_α, $\alpha = 1, 2$, be an essential surface in M such that $\partial F_\alpha \cap T$ has $n_\alpha > 0$ components. These all have slope r_α, say, on T. Let $\Delta = \Delta(r_1, r_2)$. By standard arguments, we may isotope F_1, say, so that F_1 and F_2 intersect transversely in a finite disjoint union of circles and properly embedded arcs, where each circle is essential in F_α and no arc with both its endpoints on T is parallel in F_α to a subarc of ∂F_α, $\alpha = 1, 2$, and each component of $\partial F_1 \cap T$ meets each component of $\partial F_2 \cap T$ in Δ points.

If A is an arc component of $F_1 \cap F_2$ with some endpoint on T, then, considering A as it lies in F_α, we label that endpoint with the (number of the) corresponding component of $\partial F_\beta \cap T$. (Here and throughout the paper we use the convention of [L] that $\{\alpha, \beta\} = \{1, 2\}$.) Thus around each component of $\partial F_\alpha \cap T$ we see the labels $1, 2, \ldots, n_\beta$ in cyclic order, this sequence being repeated Δ times. If we orient F_α, then this induces an orientation on each component a of $\partial F_\alpha \cap T$, and we assign a sign $\pm$ to a according to the direction of this orientation as a lies on T. We adopt the convention that if the sign of a is $+$, then the labels $1, 2, \ldots, n_\beta$ appear in anticlockwise order around a, and if the sign is $-$, then they appear in clockwise order. This kind of labeling was first used in [L].

An arc component of $F_1 \cap F_2$ with both endpoints on T will be called an *internal arc* of $F_1 \cap F_2$. The orientability of M then gives us the *parity rule*: if A is an internal arc of $F_1 \cap F_2$ on F_α, then the components of $\partial F_\alpha \cap T$ joined by A have

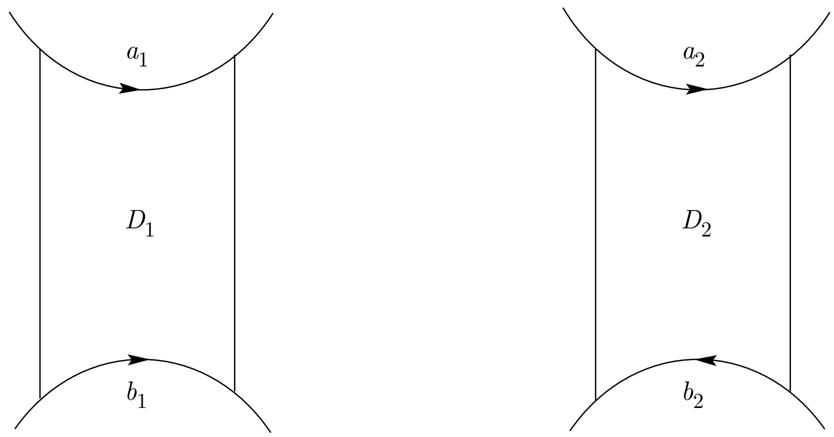


FIGURE 1

the same (resp. opposite) sign if and only if the labels at the endpoints of A have opposite (resp. the same) sign.

Let $K_{p,q} \subset \operatorname{int} S^1 \times D^2$ be a (p,q)-*curve*, $q \geq 2$, that is, a simple closed curve on the boundary of a concentric solid torus that winds around p times meridionally and q times longitudinally. The exterior of $K_{p,q}$, $S^1 \times D^2 - \operatorname{int} N(K_{p,q})$, is called a (p,q)-*cable space*, and a pair (M,T) is said to be *cabled* if M contains a (p,q)-cable space C with $C \cap \partial M = T$. It is easy to see that if $p \equiv \pm p' \pmod q$ then the corresponding cable spaces are homeomorphic.

Lemma 2.1. *Suppose that $F_1 \cap F_2$ contains a pair of internal arcs that are parallel in both F_1 and F_2. Then (M,T) is $(1,2)$-cabled.*

Proof. Let D_1 and D_2 be disks (rectangles) that realize the parallelism of arcs A and A', in F_1 and F_2 respectively. By taking innermost such disks we may suppose that $D_1 \cap D_2 = A \cup A'$. Let the arcs join components a_1, b_1 of ∂F_1 and a_2, b_2 of ∂F_2. Then a_1 and b_1 (say) are of opposite sign, whilst a_2 and b_2 are of the same sign. (See Figure 1, where the arrows indicate coherent directions on T.) A priori there are two possibilities for the way in which D_1 and D_2 are identified along A and A', illustrated by the arrows on A and A' in Figures 2(a) and (b).

In fact, case (a) is impossible. To see this, let P, Q, R, S be the points, and $\alpha_1, \beta_1, \alpha_2, \beta_2$ the arcs, indicated in Figure 2(a). Note that $P, Q \in a_1 \cap a_2$, and $R, S \in b_1 \cap b_2$. Then

$$|\alpha_1 \cap b_2| = |\beta_1 \cap a_2| = k_1 \geq 1 \text{ , and}$$
$$|\alpha_2 \cap b_1| = |\beta_2 \cap a_1| = k_2 \geq 1 \text{ .}$$

Now consider the disjoint simple closed curves $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$ on T. Orient each pair a_1, b_1 and a_2, b_2 coherently on T as in Figure 1; this orients $\alpha_1, \alpha_2, \beta_1, \beta_2$. Now give α the orientation induced by α_1 (which is equal to that induced by $-\alpha_2$), and β the orientation induced by β_1 (which is equal to that induced by β_2). Then (with some convention) we have the following algebraic intersection numbers:

$$\alpha \cdot a_1 = \alpha \cdot b_1 = (-\alpha_2) \cdot b_1 = -k_2 \text{ ,}$$
$$\beta \cdot a_1 = \beta_2 \cdot a_1 = k_2 \text{ .}$$

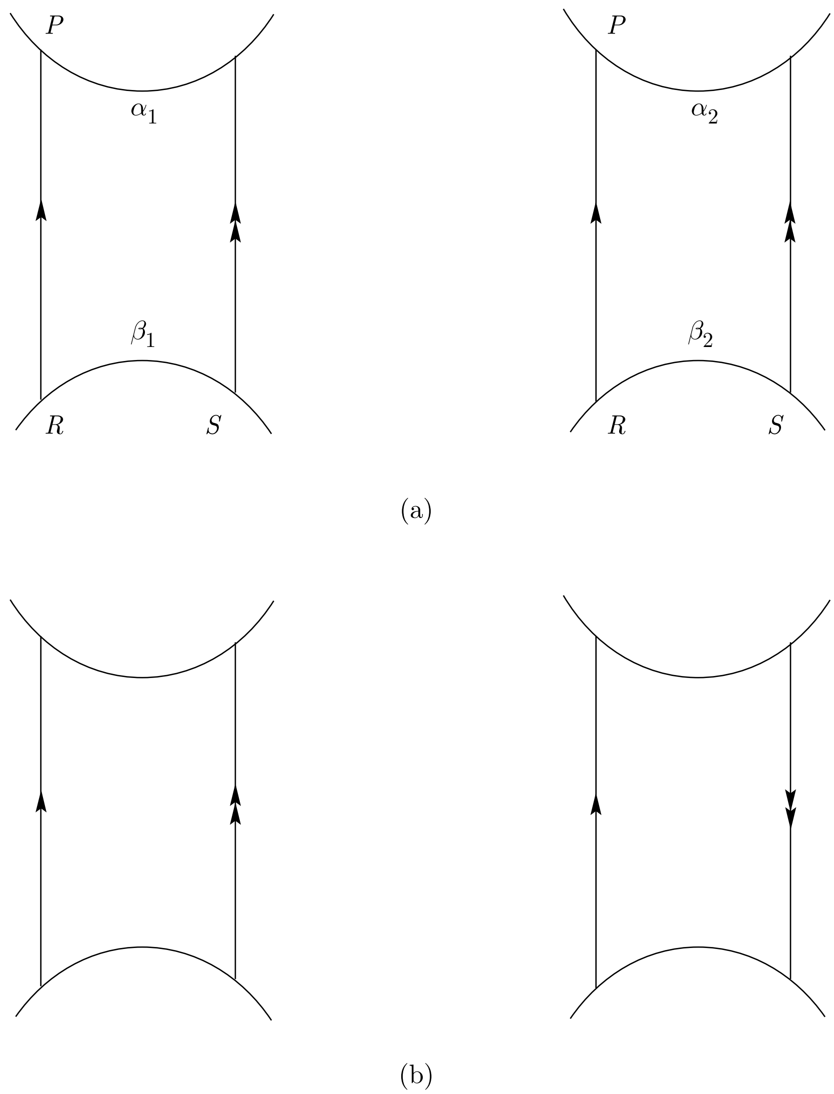


FIGURE 2

On the other hand, if $a_2 \neq b_2$, then

$$\alpha \cdot a_2 = \alpha \cdot b_2 = \alpha_1 \cdot b_2 = k_1 \ ,$$

$$\beta \cdot a_2 = \beta_1 \cdot a_2 = k_1 \ .$$

This contradicts the fact that α and β are disjoint.

Similarly, if $a_2 = b_2$, then $k_1 \geq 2$, and we have

$$\alpha \cdot a_2 = k_1 - 1 \ ,$$

$$\beta \cdot a_2 = k_1 - 1 \ .$$

This is again a contradiction.

In case (b), $D_1 \cup D_2$ is a Möbius band B with $(B, \partial B) \subset (M, T)$, from which it easily follows that (M, T) is $(1, 2)$-cabled. (See [GLi, Proposition 1.3, case (1)].) $\square$

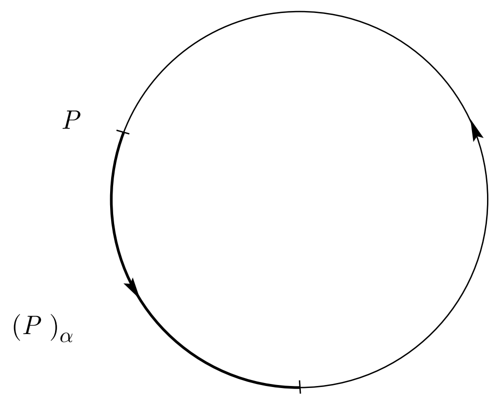


FIGURE 3

Suppose that F_α contains a family $\boldsymbol{A}$ of n_β mutually parallel internal arcs of $F_1 \cap F_2$, joining boundary components a and b (possibly equal) of F_α. Let A_i be the arc with label i at one end of $\boldsymbol{A}$, $i = 1, 2, \ldots, n_\beta$. Then A_i has label $\pi(i)$ at the other end, for some permutation π of $\{1, 2, \ldots, n_\beta\}$. Observe that π is of the form

$$\pi(i) \equiv \varepsilon i + p \qquad (\text{mod } n_\beta) \ ,$$

where $\varepsilon = \pm 1$ according as a and b are of opposite sign or of the same sign. The family $\boldsymbol{A}$ determines π up to inversion.

If $\varepsilon = +1$, then any orbit of π contains exactly $n_\beta/(n_\beta, p)$ elements, all of the same sign. If $\varepsilon = -1$, then any orbit of π contains exactly 2 elements, of opposite sign (hence n_β is even). In particular, the following lemma is immediate.

Lemma 2.2. *Suppose that F_α contains n_β mutually parallel internal arcs of $F_1 \cap F_2$ such that the corresponding permutation has only one orbit. Then either*

(1) *∂F_β has exactly two components, of opposite sign; or*
(2) *all components of ∂F_β are of the same sign.*

Let θ be an orbit of π. The arcs A_i, $i \in \theta$, define a circle $C_\theta(\boldsymbol{A})$ in $\widehat{F}_\beta$ in the obvious way. The following lemma is essentially proved in [GLi, Section 5].

Lemma 2.3. *Suppose that F_α contains a family $\boldsymbol{A}$ of n_β mutually parallel internal arcs of $F_1 \cap F_2$, such that for some orbit θ of the corresponding permutation, $C_\theta(\boldsymbol{A})$ bounds a disk in $\widehat{F}_\beta$. Then (M, T) is cabled.*

Our next goal is to prove Lemma 2.5, which gives necessary conditions for a pair of internal arcs in F_β to be parallel in F_α. This will be used repeatedly in Sections 6 through 9 to rule out certain configurations of arcs in $F_1 \cap F_2$.

Orient all the components of $\partial F_\alpha \cap T$ coherently on T, $\alpha = 1, 2$. If P and Q are two (distinct) points in some component a of $\partial F_\alpha \cap T$, let $(PQ)_\alpha$ denote the arc in a that goes from P to Q with respect to our chosen orientation of a. (See Figure 3.)

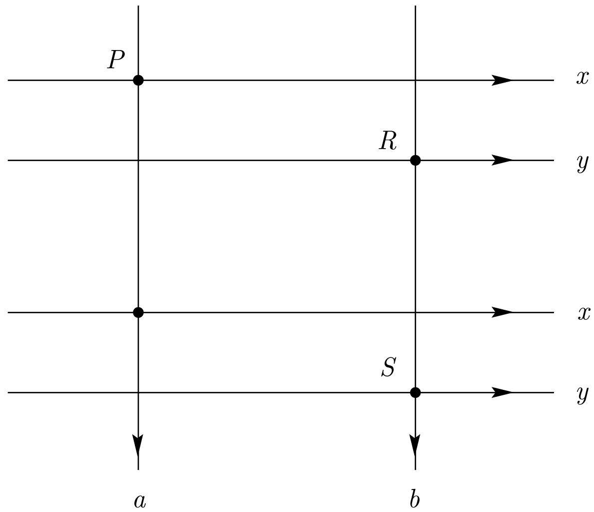


FIGURE 4

Suppose now that $P, Q \in a \cap \partial F_\beta$. Then we define

$$\tau_\alpha(P,Q) = |(PQ)_\alpha \cap \partial F_\beta| - 1.$$

In other words, $\tau_\alpha(P,Q)$ is (the number of labels on the arc $(PQ)_\alpha$) -1. Note that reversing the orientation of a sends $\tau_\alpha(P,Q)$ to $\tau_\alpha(Q,P) = \Delta n_\beta - \tau_\alpha(P,Q)$.

Lemma 2.4. *Let a, b be components of $\partial F_\alpha \cap T$ and x, y components of $\partial F_\beta \cap T$.*

(i) *Suppose that $P, Q \in a \cap x$ and $R, S \in b \cap y$. If $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$ then $\tau_\beta(P,Q) = \tau_\beta(R,S)$.*

(ii) *Suppose that $P \in a \cap x$, $Q \in a \cap y$, $R \in b \cap x$, and $S \in b \cap y$. If $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$ then $\tau_\beta(P,R) = \tau_\beta(Q,S)$.*

Proof. (i) The situation is shown in Figure 4. (Note that we allow the possibilities $a = b$, $x = y$.) Since $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$, there is a homeomorphism of $(T; \partial F_1 \cap T, \partial F_2 \cap T)$ taking P to R and Q to S. Hence $\tau_\beta(P,Q) = \tau_\beta(R,S)$.

(ii) This situation is shown in Figure 5. Again, since $\tau_\beta(P,Q) = \tau_\alpha(R,S)$ there is a homeomorphism of $(T; \partial F_1 \cap T, \partial F_2 \cap T)$ taking P to R and Q to S. Hence $\tau_\beta(P,R) = \tau_\beta(Q,S)$. □

We apply Lemma 2.4 to the situation where P, Q, R and S are the endpoints of two parallel arcs in F_α. It is sometimes convenient to express the result in terms of the following variant of τ_α. Namely, instead of orienting the components of $\partial F_\alpha \cap T$ coherently on T, give them the orientation induced by some orientation of F_α, and then define δ_α in exactly the same way as τ_α. Thus $\delta_\alpha(P,Q) = \tau_\alpha(P,Q)$ or $\tau_\alpha(Q,P)$, depending on the sign of the component a of $\partial F_\alpha \cap T$ containing P and Q.

Lemma 2.5. *Let A and B be internal arcs of $F_1 \cap F_2$ that are parallel in F_α, with endpoints P, R and Q, S respectively, labeled so that the boundary of the rectangle in F_α that realizes the parallelism is $PQSR$ (see Figure 6). Let x, y be components of $\partial F_\beta \cap T$.*

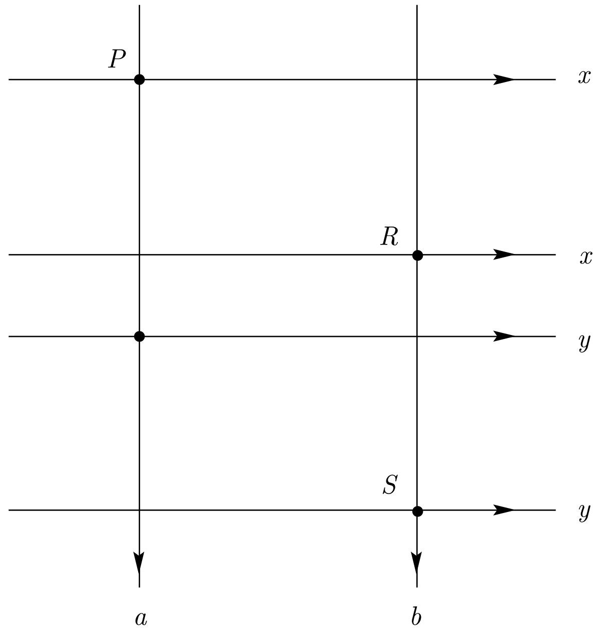


FIGURE 5

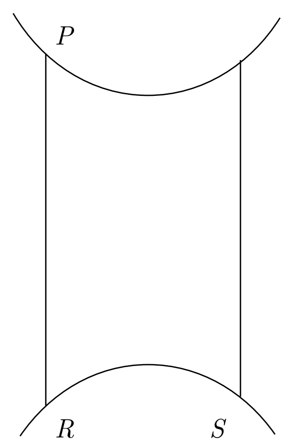


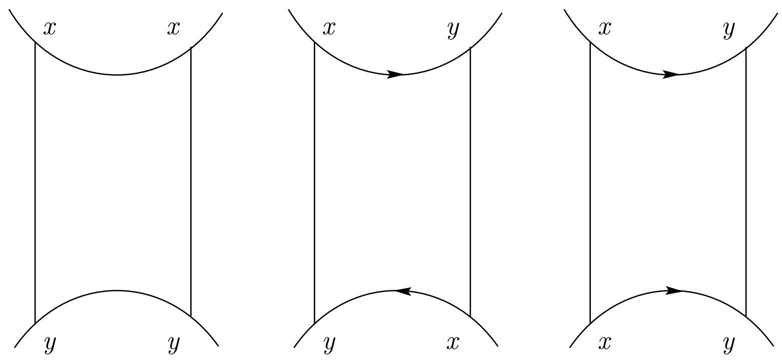
FIGURE 6

(i) *Suppose that* $P, Q \in x$ *and* $R, S \in y$ *(see Figure 7(i)). Then* $\delta_\beta(P, Q) = \delta_\beta(R, S)$.

(ii) *Suppose that* $P, S \in x$, $Q, R \in y$ *(see Figure 7(ii)), and that* x *and* y *are of opposite sign. Then* $\delta_\beta(P, S) = \delta_\beta(R, Q)$.

(iii) *Suppose that* $P, R \in x$ *and* $Q, S \in y$ *(see Figure 7(iii)). Then* $\tau_\beta(P, R) = \tau_\beta(Q, S)$.

Proof. Let a and b be the components of $\partial F_\alpha \cap T$ containing P and R respectively.



(i) (ii) (iii)

FIGURE 7

(i) First suppose that a and b are of the same sign. Then $\tau_\alpha(P,Q) = \tau_\alpha(S,R)$. Therefore, by Lemma 2.4(i), $\tau_\beta(P,Q) = \tau_\beta(S,R)$. Since x and y are of opposite sign, this implies that $\delta_\beta(P,Q) = \delta_\beta(R,S)$.

If a and b are of opposite sign, then $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$, and hence $\tau_\beta(P,Q) = \tau_\beta(R,S)$. Since x and y are now of the same sign, we get $\delta_\beta(P,Q) = \delta_\beta(R,S)$ as before.

(ii) Since x and y are of opposite sign, a and b are of the same sign. Therefore $\tau_\alpha(P,Q) = \tau_\alpha(S,R)$, and hence, by Lemma 2.4(ii), $\tau_\beta(P,S) = \tau_\beta(Q,R)$, giving $\delta_\beta(P,S) = \delta_\beta(R,Q)$.

(iii) Here, since both endpoints of A lie on the same component of $\partial F_\beta \cap T$ (and similarly for B), a and b are necessarily of opposite sign. Therefore $\tau_\alpha(P,Q) = \tau_\alpha(R,S)$. By Lemma 2.4(ii), this implies that $\tau_\beta(P,R) = \tau_\beta(Q,S)$. □

In practice, it is convenient to express Lemma 2.5 as follows. Let A be an internal arc of $F_1 \cap F_2$ in F_β, whose endpoints have label a at boundary component x and label b at boundary component y. If $x \neq y$, then we can use A^x to unambiguously denote the endpoint of A at x. Similarly, if $a \neq b$, then we can use $A(a)$ to unambiguously denote the endpoint of A with label a. (Note that we can't have both $x = y$ and $a = b$, by the parity rule.)

Lemma 2.5 asserts that if A and B are internal arcs of $F_1 \cap F_2$ in F_β that are parallel in F_α then:

in case (ii), and case (i) when $x \neq y$, $\delta_\beta(A^x, B^x) = \delta_\beta(A^y, B^y)$;
in case (i) when $a \neq b$, $\delta_\beta(A(a), B(a)) = \delta_\beta(A(b), B(b))$;
in case (iii) (where necessarily $a \neq b$), $\tau_\beta(A(a), A(b)) = \tau_\beta(B(a), B(b))$.

We conclude this section with an observation about the points of intersection of ∂F_1 and ∂F_2 on T which will be useful in the sequel.

Let a_α be a component of $\partial F_\alpha \cap T$, $\alpha = 1, 2$. Parametrize a_1 as $\mathbb{R}/\mathbb{Z}$ in such a way that the points of $a_1 \cap a_2$ are $\{i/\Delta : i = 1, 2, \dots, \Delta\}$. Then a_2 may be parametrized as $\mathbb{R}/\mathbb{Z}$ so that the points of $a_1 \cap a_2$ have (a_1, a_2)-coordinates $\{(i/\Delta,\, di/\Delta) : i = 1, 2, \dots, \Delta\}$ for some integer $d = d_{12}$ coprime to Δ. By re-orienting a_2 if necessary we may assume that $1 \leq d \leq \Delta/2$. Note that $d_{21} \equiv \pm d_{12}^{-1} \pmod{\Delta}$.

3. Graphs in surfaces

Given surfaces F_1, F_2 in M as described in Section 2, it is convenient to regard the arcs of $F_1 \cap F_2$ as defining a graph Γ_α in $\widehat{F}_\alpha$, $\alpha = 1, 2$. Thus we regard the disks $C\ell(\widehat{F}_\alpha - F_\alpha)$ as the "fat" *vertices* of Γ_α, and the arcs of $F_1 \cap F_2$ with at least one endpoint on T as the *edges* of Γ_α. We call an edge of Γ_α corresponding to an internal arc an *internal edge*, and an edge corresponding to an arc with exactly one endpoint on T (and the other on $\partial\widehat{F}_\alpha$) a *boundary edge*. Since no internal arc of $F_1 \cap F_2$ is boundary-parallel in F_α, no face of Γ_α is a disk with one side.

In the rest of the paper we shall use the topological and graph-theoretic terminologies interchangeably, for instance sometimes referring to arcs of $F_1 \cap F_2$ in F_α and boundary components of F_α, and sometimes to edges and vertices of Γ_α.

The next two lemmas are about graphs in tori. The first is a special case of [GLi, Lemma 6.2].

Lemma 3.1. *Let Γ be a graph in a torus such that no face of Γ is a disk with one side. Suppose that, for some positive integer n, the valency of each vertex of Γ is greater than $6n$. Then Γ has $n+1$ mutually parallel edges.*

If Γ is a graph in a surface F as described above, then the *reduced graph* of Γ is the graph $\overline{\Gamma}$ in F obtained by amalgamating each set of mutually parallel edges of Γ to a single edge.

Lemma 3.2. *Let Γ be a graph in a torus such that no face of Γ is a disk with one side, and such that, for some positive integer n, each vertex of Γ has valency $6n$. Suppose that Γ does not contain $n+1$ mutually parallel edges. Then each parallelism class of edges of Γ has n members, and each face of the reduced graph of Γ is a disk with 3 sides.*

Proof. Let $\overline{\Gamma}$ be the reduced graph of Γ, and let V and E be the number of vertices and edges of $\overline{\Gamma}$.

Each vertex of $\overline{\Gamma}$ has valency $\geq (6n/n) = 6$. Therefore $2E \geq 6V$, giving $V \leq E/3$.

Let F be the number of disk faces of $\overline{\Gamma}$. Then, since each disk face of $\overline{\Gamma}$ has at least 3 sides, $2E \geq 3F$, giving $F \leq 2E/3$.

Finally, we have

$$V - E + \sum \chi(f) = \chi(\text{torus}) = 0 \ ,$$

where $\sum \chi(f)$ is summed over all faces f of $\overline{\Gamma}$.

Hence

$$0 = V - E + \sum \chi(f) \leq V - E + F \leq \frac{E}{3} - E + \frac{2E}{3} = 0 \ .$$

It follows that the above inequalities are all equalities. Thus each vertex of $\overline{\Gamma}$ has valency 6, each face of $\overline{\Gamma}$ is a disk, and $2E = 3F$. □

We shall need the following lemma in Section 12.

Lemma 3.3. *Let Γ be a graph in an annulus such that no face of Γ is a disk with one side, and such that, for some positive integer n, each vertex of Γ has valency at least $6n$. Then Γ has either $n+1$ mutually parallel internal edges or $2n$ mutually parallel boundary edges.*

Proof. Let $\overline{\Gamma}$ be the reduced graph of Γ, and let V, E, and F be the number of vertices, edges, and disk faces of $\overline{\Gamma}$. Let $\Sigma = \sum \chi(f)$, summed over all faces of $\overline{\Gamma}$. Suppose, for a contradiction, that Γ has neither $n+1$ parallel internal edges nor $2n$ parallel boundary edges.

First suppose that Γ has no boundary edges. The valency of each vertex of $\overline{\Gamma}$ is at least $(6n/n) = 6$. Hence $2E \geq 6V$, giving $V \leq E/3$. Also, $2E \geq 3F$, giving $F \leq 2E/3$. Since $\overline{\Gamma}$ has a non-disk face (containing a boundary component of the annulus), we have $\Sigma < F$. Therefore

$$0 = \chi(\text{annulus}) = V - E + \Sigma < V - E + F \leq E\big(\frac{1}{3} - 1 + \frac{2}{3}\big) = 0 \ ,$$

a contradiction.

Now suppose that Γ has a least one boundary edge. For each vertex v of Γ, let $\alpha(v)$ (resp. $\beta(v)$) be the number of incidences of internal (resp. boundary) edges at v, and let $\bar{\alpha}(v), \bar{\beta}(v)$ be the corresponding quantities for $\overline{\Gamma}$. By hypothesis, $\alpha(v) + \beta(v) \geq 6n$, for all v. Also by hypothesis, $\bar{\alpha}(v) \geq \alpha(v)/n$ and $\bar{\beta}(v) \geq \beta(v)/(2n-1)$, for all v. Since $\sum \beta(v) \neq 0$ (the sum being taken over all vertices of Γ), we have

$$\textstyle\sum \bar{\beta}(v) \geq \dfrac{\sum \beta(v)}{2n-1} > \dfrac{\sum \beta(v)}{2n} \ .$$

Hence

$$2E = \textstyle\sum \bar{\alpha}(v) + 2\sum \bar{\beta}(v) > \dfrac{\sum \alpha(v) + \sum \beta(v)}{n} \geq \dfrac{6nV}{n} = 6V \ .$$

Thus $V > E/3$. Also, $2E \geq 3F$, giving $F \leq 2E/3$. Therefore

$$0 = V - E + \Sigma \leq V - E + F < E\big(\frac{1}{3} - 1 + \frac{2}{3}\big) = 0 \ ,$$

another contradiction. □

4. Intersections of punctured tori; the generic case

From now on, through Section 11, we will assume that $\widehat{F}_1$ and $\widehat{F}_2$ are tori, and that $\Delta = \Delta(r_1, r_2) \geq 6$.

The following lemma is an immediate consequence of Lemma 3.1.

Lemma 4.1. *If $\Delta > 6$ then F_α contains $n_\beta + 1$ mutually parallel arcs of $F_1 \cap F_2$.*

This motivates the following definitions. We say that F_α is *excellent* if it contains $n_\beta + 1$ mutually parallel arcs of $F_1 \cap F_2$ such that the corresponding permutation has more than one orbit. We say that F_α is *good* if it contains $n_\beta + 1$ mutually parallel arcs of $F_1 \cap F_2$ but the corresponding permutation has only one orbit. We say that F_α is *bad* if it does not contain $n_\beta + 1$ mutually parallel arcs of $F_1 \cap F_2$.

The generic situation is taken care of by the following lemma.

Lemma 4.2. *If F_α is excellent then (M, T) is cabled.*

Proof. Without loss of generality we may assume that the labels at one end of the parallel family of arcs are $1, 2, \ldots, n_\beta, 1$. Let the arcs be $A_1, A_2, \ldots, A_{n_\beta}, B_1$, where the subscript indicates the label at that end. (See Figure 8.)

The set of arcs $\boldsymbol{A} = \{A_1, A_2, \ldots, A_{n_\beta}\}$ determines a permutation π. Let θ be the orbit of π containing 1. By hypothesis there exists another orbit φ. By Lemma 2.3 we may assume that the circles $C_\theta(\boldsymbol{A})$ and $C_\varphi(\boldsymbol{A})$ are essential in the torus $\widehat{F}_\beta$.

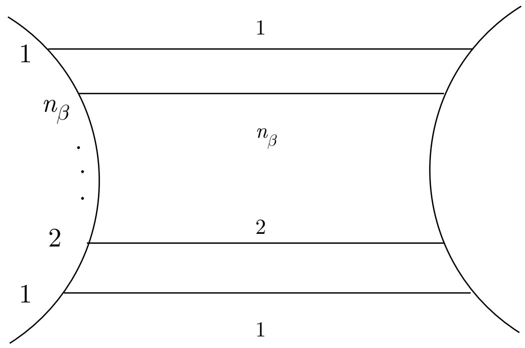


FIGURE 8

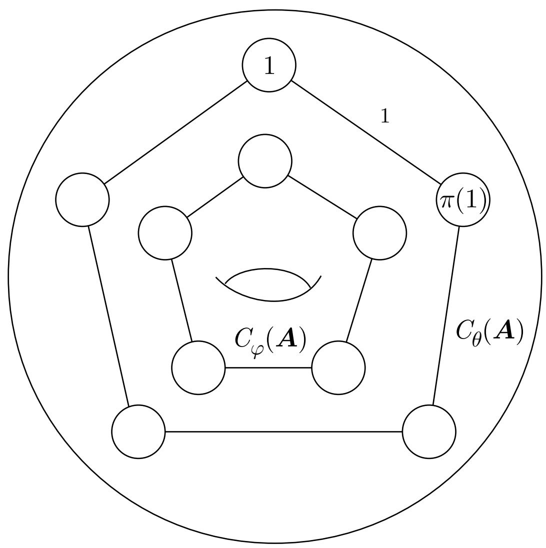


FIGURE 9

They are therefore parallel in $\widehat{F}_\beta$. Now consider the arc B_1. Since it is disjoint from $C_\varphi(\boldsymbol{A})$, either it is parallel to A_1 in F_β, or, letting $\boldsymbol{B}$ be the set of parallel arcs $\{A_2, \ldots, A_{n_\beta}, B_1\}$, we have that $C_\theta(\boldsymbol{B})$ bounds a disk in $\widehat{F}_\beta$. (See Figure 9.) The result then follows from Lemmas 2.1 and 2.3 respectively. □

We will additionally assume from now on (through Section 11) that (M,T) is not cabled. Then, after Lemmas 4.1 and 4.2, exactly one of the following must hold:

(I) F_1 and F_2 are good;
(II) $\Delta = 6$, F_1 (say) is good and F_2 is bad;
(III) $\Delta = 6$, F_1 and F_2 are bad.

It is convenient to organize the rest of the proof along rather different lines, however, bearing in mind Lemma 2.2. Specifically, we consider the following cases:

(A) $n_1 = n_2 = 2$, and the two boundary components of F_α have opposite sign, $\alpha = 1, 2$.

(B) $n_1 = 2$, the two boundary components of F_1 have opposite sign, and all boundary components of F_2 have the same sign.

Case (B) is subdivided into four subcases as follows:

(B)(1) $n_2 = 1$,
(B)(2) $n_2 = 2$,
(B)(3) $n_2 = 3$,
(B)(4) $n_2 \geq 4$.

(C) $\Delta = 6$, $n_1 \geq 3$, $n_2 = 2$, the two boundary components of F_2 have opposite sign, and F_2 is bad.

(D) $\Delta = 6$, $n_1 \geq 3$, all boundary components of F_2 have the same sign, and F_2 is bad.

(D) is subdivided into three subcases:

(D)(1) $n_2 = 1$,
(D)(2) $n_2 = 2$,
(D)(3) $n_2 \geq 3$.

(E) $\Delta = 6$, F_1 and F_2 bad.

By Lemma 2.2, if (I) holds then we're in either case (A) or (B), and if (II) holds then we're in either case (A) or (B) or (C) or (D).

It will turn out that the possibilities for $(F_1, F_2; F_1 \cap F_2)$ in each of these cases are as follows:

Case (A): one example with $\Delta = 6$ and two examples with $\Delta = 8$;

Case (B)(1): one example, with $\Delta = 7$.

The combinatorics allow one example in Case (B)(1) with $\Delta = 6$, and an infinite family of examples in Case (D)(1) with $\Delta = 6$, but these will be shown to be topologically degenerate in Section 11.

All other cases are impossible.

Perhaps surprisingly, the hardest case to deal with appears to be (B), especially cases (B)(3) and (B)(4).

5. The non-generic case; preparatory lemmas

In Lemmas 5.1 and 5.2 we explicitly identify the reduced graph $\overline{\Gamma}_\alpha$ in the cases where F_α has one or two boundary components. Next, in Lemma 5.3, we establish some additional properties of the graph Γ_α in the case where F_1 and F_2 each have two boundary components. These will be needed in Sections 6 and 7. Finally, we note some properties of the graph Γ_β in the case where Γ_α contains n_β parallel edges such that the corresponding permutation has only one orbit. These are stated in Lemma 5.4. This, and Corollary 5.5, will be used in Sections 7 and 9.

Throughout, we regard two graphs in a surface as equivalent if there is a homeomorphism of the surface taking one to the other.

Lemma 5.1. *If F_α has a single boundary component, then the reduced graph $\overline{\Gamma}_\alpha$ is a subgraph of the graph illustrated in Figure 10.*

Proof. This follows immediately from the fact that the edges of $\overline{\Gamma}_\alpha$ are non-parallel essential loops in $\widehat{F}_\alpha$. □

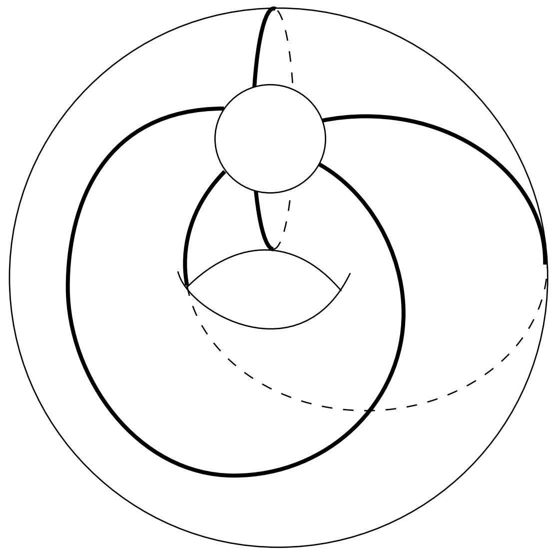


FIGURE 10

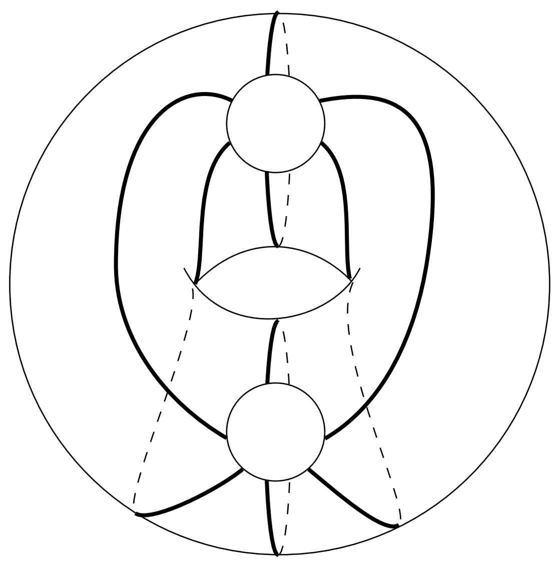


FIGURE 11

Lemma 5.2. *If F_α has exactly two boundary components, then the reduced graph $\overline{\Gamma}_\alpha$ is a subgraph of the graph illustrated in Figure 11.*

Proof. Let the vertices of Γ_α be a and b. Since a and b have the same valency, the number of loops at a is equal to the number of loops at b.

If this number is non-zero, there is exactly one loop at each vertex in $\overline{\Gamma}_\alpha$. Cutting $\widehat{F}_\alpha$ along one of these loops (at b, say) gives an annulus, as illustrated in Figure 12. Then $\overline{\Gamma}_\alpha$ must be a subgraph of the graph illustrated in Figure 13, as desired.

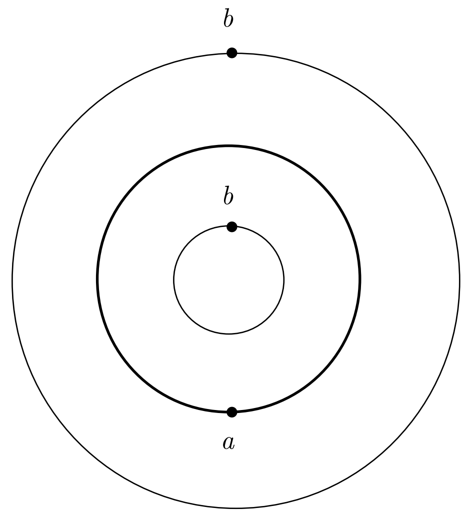


FIGURE 12

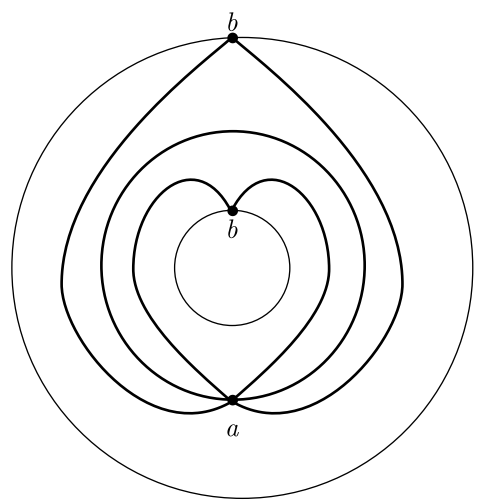


FIGURE 13

If $\overline{\Gamma}_\alpha$ has no loops, but two non-parallel edges, then cutting $\widehat{F}_\alpha$ along these two edges gives an annulus as illustrated in Figure 14. Then clearly $\overline{\Gamma}_\alpha$ is a subgraph of either the graph illustrated in Figure 15(i) or the graph illustrated in Figure 15(ii). But these both correspond to Figure 11 (with the loops removed). □

If Γ is a graph in a torus such that $\overline{\Gamma}$ is a subgraph of the graph illustrated in Figure 10, then Γ is determined by the triple $(\beta_1, \beta_2, \beta_3)$ of non-negative integers that records the number of edges in each parallelism class, as shown in Figure 16. We say $\Gamma \cong H(\beta_1, \beta_2, \beta_3)$. Note that $H(\beta_1, \beta_2, \beta_3)$ is invariant under any permutation of the β's.

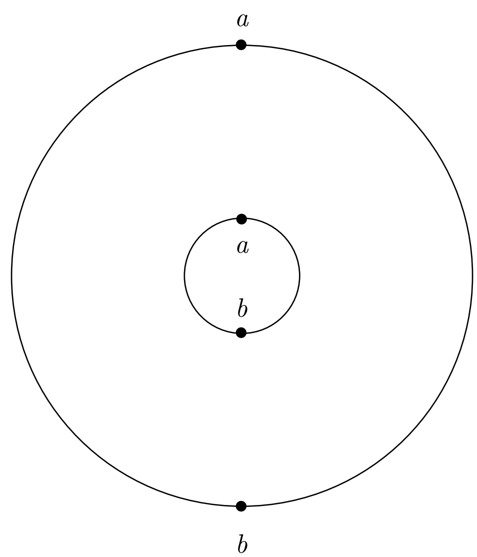


FIGURE 14

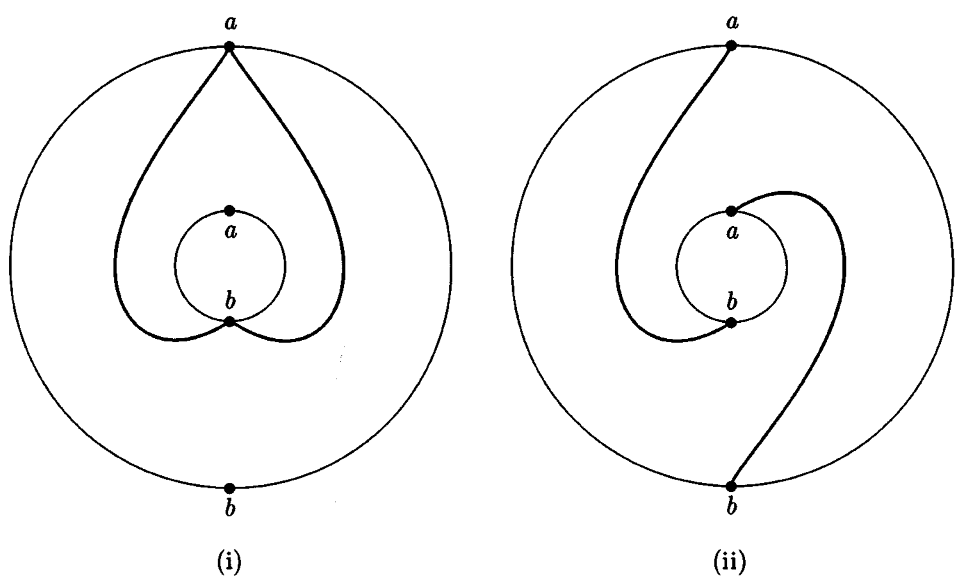


FIGURE 15

Similarly, if $\overline{\Gamma}$ is a subgraph of the graph in Figure 11, then Γ is determined by a quintuple $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of non-negative integers, as shown in Figure 17. In this case we say $\Gamma \cong G(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We abbreviate $G(0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to $G(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Note that

$$G(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \cong G(\alpha_0, \alpha_3, \alpha_4, \alpha_1, \alpha_2) \cong G(\alpha_0, \alpha_4, \alpha_3, \alpha_2, \alpha_1)$$
$$\cong G(\alpha_0, \alpha_2, \alpha_1, \alpha_4, \alpha_3).$$

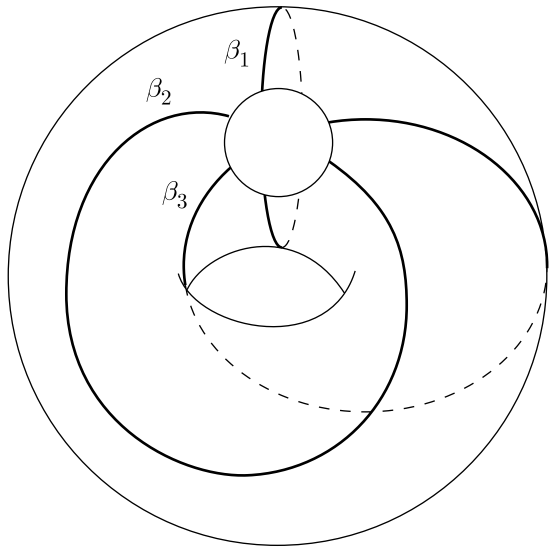


FIGURE 16

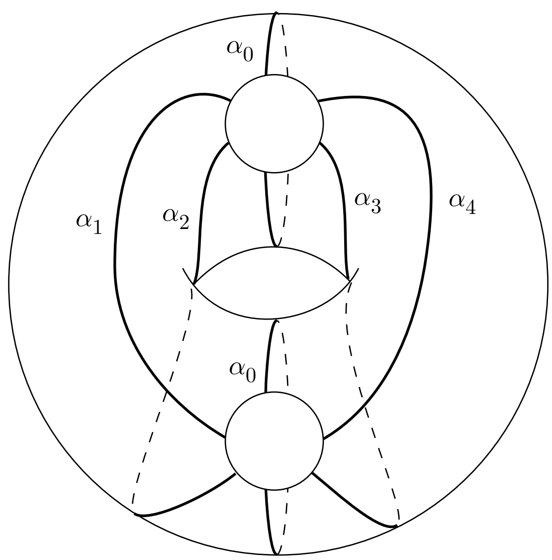


FIGURE 17

In addition, $G(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cong G(\alpha_2, \alpha_3, \alpha_4, \alpha_1)$.

For the next lemma, assume that $n_1 = n_2 = 2$. Thus $\Gamma_1 \cong G(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and similarly for Γ_2.

A parallel family of edges of Γ_1 corresponds to either loops in Γ_2, or edges in Γ_2 with distinct endpoints. Define ε_i to be 0 or 1 according as the parallel family labelled by α_i is of the first or second kind, $i = 0, 1, 2, 3, 4$. Note that $\varepsilon_0 = 1$.

Lemma 5.3. *Suppose that $n_1 = n_2 = 2$. Then*

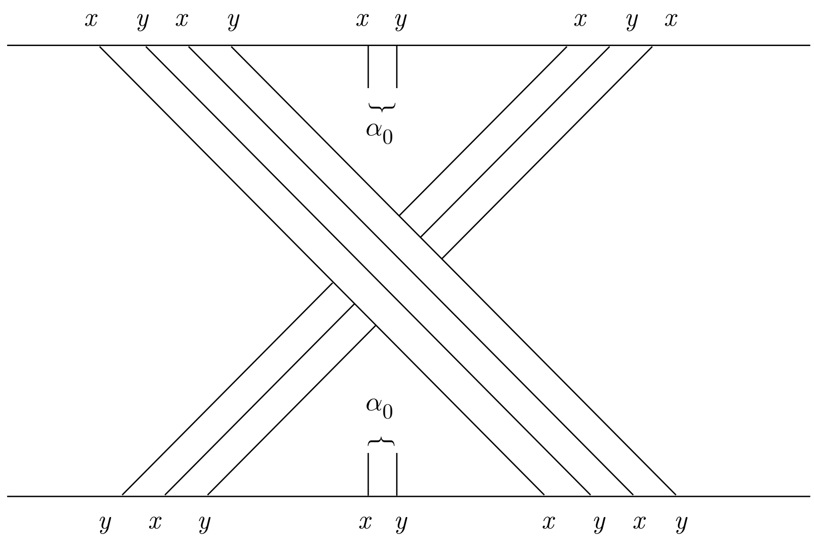


FIGURE 18

(i) *if* $\varepsilon_i = 0$ *then* $\alpha_i \leq 2$;
(ii) *if* $\varepsilon_i = 1$ *then* $\alpha_i \leq 4$;
(iii) $\alpha_i + \varepsilon_i \equiv \alpha_j + \varepsilon_j \pmod 2$, $i, j = 1, 2, 3, 4$.

Proof. (i) and (ii) follow respectively from the facts that Γ_2 has at most 2 parallelism classes of loops, and at most 4 parallelism classes of edges with distinct endpoints, together with Lemma 2.1.

To prove (iii), first consider the parallelism classes of edges of Γ_1 corresponding to $i = 2$ and 3. The way the endpoints of the corresponding arcs lie on the two boundary components of F_1 is illustrated in Figure 18. Let the boundary components of F_2 be x and y. Since the labels x, y alternate around the boundary components of F_1, the fact that $\alpha_2 + \varepsilon_2 \equiv \alpha_3 + \varepsilon_3 \pmod 2$ easily follows by inspection. (E.g., Figure 18 explicitly illustrates the case $\alpha_2 \equiv 0 \pmod 2$, $\varepsilon_2 = 0$, $\alpha_3 = 1 \pmod 2$, $\varepsilon_1 = 1$, when α_0 is even.) Similarly $\alpha_4 + \varepsilon_4 \equiv \alpha_1 + \varepsilon_1 \pmod 2$.

The arrangement of the parallelism classes of edges corresponding to $i = 1$ and 2 corresponds to Figure 18 with $\alpha_0 = 0$. Hence we get $\alpha_1 + \varepsilon_1 \equiv \alpha_2 + \varepsilon_2 \pmod 2$. Similarly $\alpha_3 + \varepsilon_3 \equiv \alpha_4 + \varepsilon_4 \pmod 2$. □

Let Γ be a graph in a torus and let x, y be vertices of Γ. Then $\nu(x, y)$ will denote the number of edges in the reduced graph $\overline{\Gamma}$ that join x and y.

Lemma 5.4. *Suppose that* Γ_α *contains* n_β *parallel edges such that the corresponding permutation* π *has only one orbit. Then*

(i) $\nu(x, \pi(x)) \leq 5$ *for all vertices* x *of* Γ_β;
(ii) *if* $\nu(x, \pi(x)) \geq 3$, *then* $\nu(y, \pi(y)) = 1$ *for* $y \neq x$, $\pi(x)$ *or* $\pi^{-1}(x)$.

Proof. Cutting the torus $\widehat{F}_\beta$ along the edges of Γ_β corresponding to the n_β parallel edges in Γ_α gives an annulus N, by Lemma 2.3 (see Figure 19).

(i) We consider how any additional edges of $\overline{\Gamma}_\beta$ that join x and $\pi(x)$ can lie in N. Such an edge either has both endpoints on the same component of ∂N (type I), or joins the two components of ∂N (type II). There are at most two edges of type I

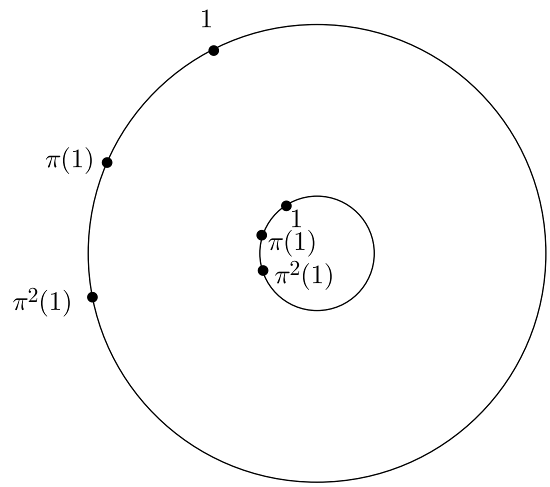


FIGURE 19

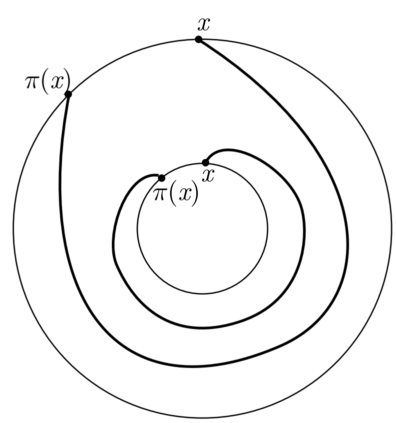


FIGURE 20

(one for each component of ∂N); see Figure 20. Also there are at most two of type II; see Figures 21(i) and 21(ii) for the only two possibilities, up to symmetry. Hence $\nu(x, \pi(x)) \leq 5$.

(ii) Note that since the assertion is vacuously true if $n_\beta \leq 3$, we may assume $n_\beta \geq 4$. Suppose $\nu(x, \pi(x)) \geq 3$. Then we have two edges of $\overline{\Gamma}_\beta$ joining x and $\pi(x)$ in the annulus N, as described in (i). Up to symmetry, the only possibilities for such a pair of edges are illustrated in Figures 20, 21 and 22. In each case it is clear that if $\{y, \pi(y)\} \cap \{x, \pi(x)\} = \emptyset$ then $\nu(y, \pi(y)) = 1$. $\square$

Corollary 5.5. *Suppose that $n_\beta \geq 4$. Then the number of edges in any parallelism class in Γ_α is at most $2n_\beta$.*

Proof. Suppose that Γ_α contains $2n_\beta + 1$ mutually parallel edges. Without loss of generality, assume that the labels at one end of the parallelism class are $1, 2, \ldots, n_\beta$, $1, 2, \ldots, n_\beta, 1$. By Lemma 4.2 the corresponding permutation π has only one orbit.

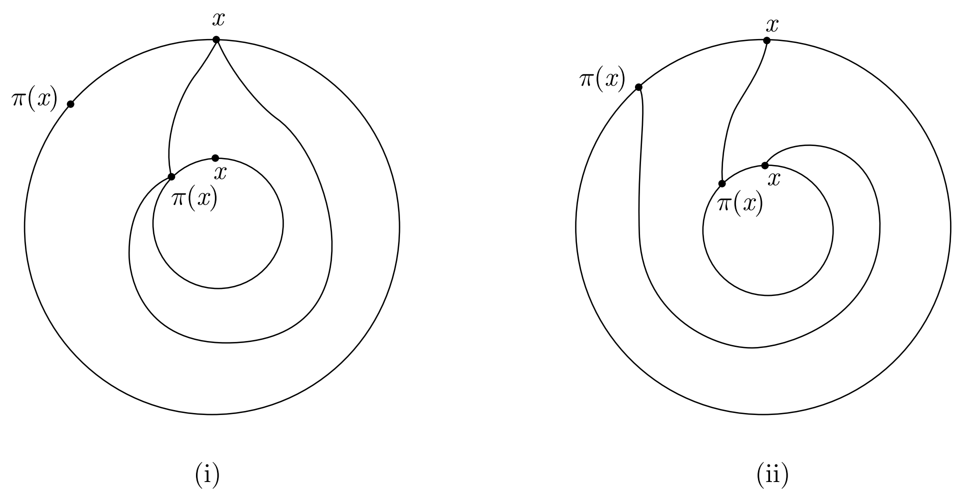


FIGURE 21

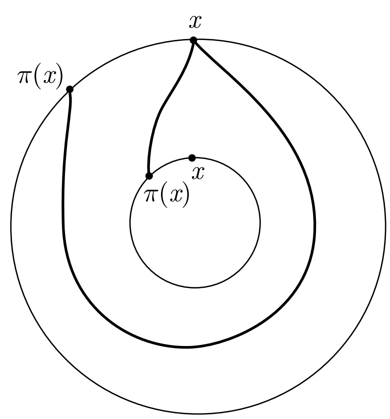


FIGURE 22

By Lemma 2.1, $\nu(x, \pi(x)) \geq 2$ for all vertices x of Γ_β, and $\nu(1, \pi(1)) \geq 3$. But since $n_\beta \geq 4$, there exists x such that $x \neq 1$, $\pi(1)$, or $\pi^{-1}(1)$. We thus get a contradiction to Lemma 5.4(ii). □

6. CASE (A)

In this section we treat the case where $n_1 = n_2 = 2$, and the two boundary components of F_α have opposite sign, $\alpha = 1, 2$. Call the boundary components of F_α (equivalently, the vertices of Γ_α) $+$ and $-$, $\alpha = 1, 2$.

By Lemma 5.2, $\Gamma_1 \cong G(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and $\Gamma_2 \cong G(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$.

Lemma 6.1. (i) $2\alpha_0 + \sum_{i=1}^{4} \alpha_i = 2\beta_0 + \sum_{i=1}^{4} \beta_i = 2\Delta$;

(ii) $2\alpha_0 = \sum_{i=1}^{4} \beta_i, \quad 2\beta_0 = \sum_{i=1}^{4} \alpha_i$ *;*

(iii) $\alpha_0, \beta_0 \leq 4$ *;*

(iv) $\alpha_i, \beta_i \leq 2, \quad i = 1, 2, 3, 4$ *.*

(v) $\alpha_i \equiv \alpha_j \pmod 2, \quad i, j = 1, 2, 3, 4,$ *and similarly for the* β*'s.*

Proof. (i) follows by considering the total number of edges of Γ_α.

By the parity rule, loops in Γ_α correspond precisely to edges in Γ_β with distinct endpoints. This gives (ii).

(iii) and (iv) follow from Lemma 5.3, (i) and (ii).

(v) follows from Lemma 5.3 (iii), since here $\varepsilon_i = 0$, $i = 1, 2, 3, 4$. □

Corollary 6.2. *Either* $\alpha_0 \geq \Delta/2$ *or* $\beta_0 \geq \Delta/2$.

Proof. Lemma 6.1, (i) and (ii). □

Corollary 6.3. $\Delta \leq 8$.

Proof. Corollary 6.2 and Lemma 6.1 (iii). □

Lemma 6.4. (i) *If* $\Delta = 8$ *then* $\Gamma_1 \cong \Gamma_2 \cong G(4, 2, 2, 2, 2)$.

(ii) $\Delta = 7$ *is impossible.*

(iii) *If* $\Delta = 6$ *then* Γ_1 *(say)* $\cong G(4, 2, 0, 0, 2)$ *and* $\Gamma_2 \cong G(2, 2, 2, 2, 2)$.

Proof. (i) By Corollary 6.2 and Lemma 6.1 (iii), α_0 (say) $= 4$. Hence by Lemma 6.1, (i) and (iv), $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2$. Lemma 6.1 (ii) (and (iv)) now gives the same conclusion for the β's.

(ii) By Corollary 6.2 and Lemma 6.1 (iii), α_0 (say) $= 4$. Hence $\beta_0 = 3$ (by Lemma 6.1, (i) and (ii)), and so (by Lemma 6.1 (iv)) $\Gamma_2 \cong G(3, 2, 2, 2, 2)$.

Let A, B, C, D be the edges of Γ_2 indicated in Figure 23. These are the 4 loops at the vertex $+$ (say) in Γ_1, and hence they are all parallel in Γ_1. But this is impossible by Lemma 2.5. More precisely, if X is an edge of Γ_2 joining vertices $+$ and $-$, let X^+, X^- denote the endpoints of X at $+$ and $-$ respectively. If two such edges X, Y are parallel in Γ_1, then by either (i) or (ii) of Lemma 2.5 we must have $\delta_2(X^+, Y^+) = \delta_2(X^-, Y^-)$. One readily verifies that the only pairs of edges among A, B, C, and D permitted to be parallel in Γ_1 by this condition are A, B and C, D. For example, for suitable choice of orientation of F_2, we have $\delta_2(A^+, C^+) = 8$, while $\delta_2(A^-, C^-) = 6$.

(iii) By Corollary 6.2 and Lemma 6.1 (iii), α_0 (say) $= 3$ or 4. We consider these two cases separately.

Case 1. $\alpha_0 = 3$. By Lemma 6.1, (i), (iv), and (v), $\Gamma_1 \cong G(3, 2, 2, 2, 0)$. Let A, B, C be the edges of Γ_1 indicated in Figure 24. These are loops at the vertex $+$ (say) in Γ_2, and hence are parallel in Γ_2. But as in (ii) above, it is easy to check that C cannot be parallel to A or B in Γ_2 by Lemma 2.5(i) or (ii).

Case 2. $\alpha_0 = 4$. By Lemma 6.1, (i), (ii), (iv) and (v), $\Gamma_1 \cong G(4, 2, 0, 0, 2)$, $G(4, 2, 2, 0, 0)$, or $G(4, 1, 1, 1, 1)$. We shall show that the last two possibilities cannot occur.

First note that by Lemma 6.1, (i) and (ii), $\Gamma_2 \cong G(2, 2, 2, 2, 2)$. Let $A_1, A_2, \ldots,$ A_6 be the edges of Γ_2 with label $+$ at vertex $+$ of Γ_2, numbered cyclically around that vertex in such a way that A_1 and A_4 are loops (see Figure 25). Since $d = d_{12} = d_{21} = 1$ (see Section 2), these edges appear in the same order around vertex $+$ of Γ_1.

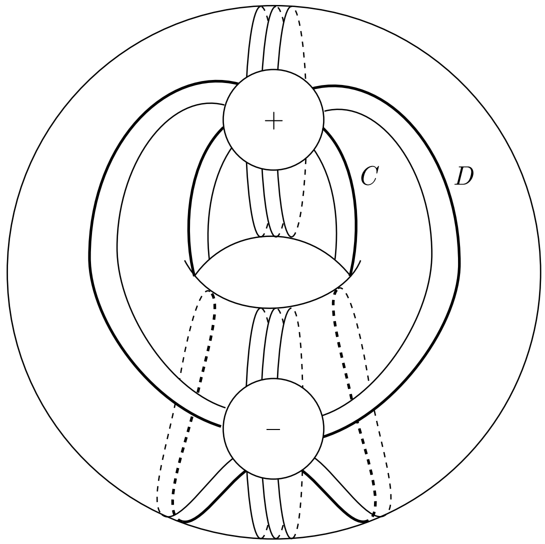


FIGURE 23

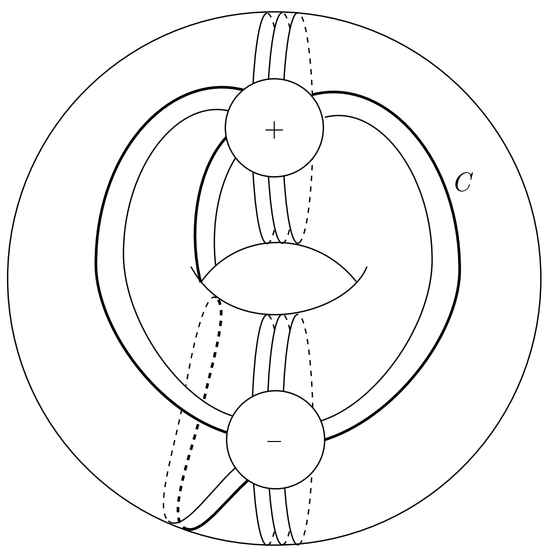


FIGURE 24

Since A_1, A_4 join distinct vertices of Γ_1, this implies that $\alpha_1 + \alpha_2 \neq 0 \neq \alpha_3 + \alpha_4$. This rules out the possibility that $\Gamma_1 \cong G(4, 2, 2, 0, 0)$.

To rule out the possibility $\Gamma_1 \cong G(4, 1, 1, 1, 1)$, consider the loops in Γ_1. The loops at vertex $+$ correspond to the edges A_2, A_3, A_5, A_6 of Γ_2. Note that each of these edges is parallel to another edge of Γ_2 that corresponds to a loop in Γ_1 at vertex $-$. Let X be the loop in Γ_1 at vertex $+$ shown in Figure 26. Let $X(\pm)$ be the endpoint of X with label $\pm$. Then (for suitable choice of orientation of boundary component $+$ of F_1) we see that $\tau_1(X(+), X(-)) = 3$. Let Y be the edge of Γ_2

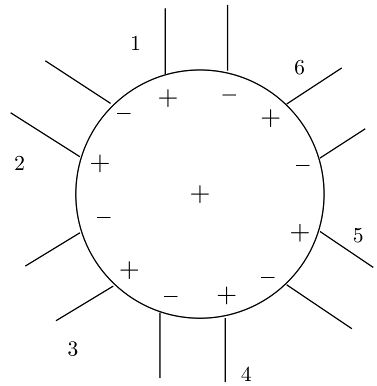


FIGURE 25

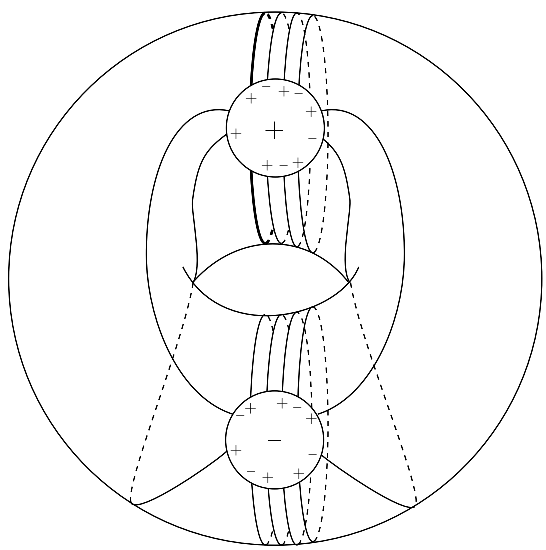


FIGURE 26

that is parallel to X in Γ_2. Then Y is a loop in Γ_1 at vertex $-$. But, recalling that boundary components $+$ and $-$ of F_1 are of opposite sign, it is easy to verify that no such loop has $\tau_1(Y(+), Y(-)) = 3$. This contradicts Lemma 2.5(iii). $\square$

The possibilities allowed by Lemma 6.4 do in fact occur. We describe the identification between the edges of Γ_1 and Γ_2 in these cases.

First consider the case given in Lemma 6.4(iii), with $\Delta = 6$. Let $A_1, A_2, \ldots, A_6$ be the edges of Γ_2 with label $+$ at vertex $+$ of Γ_2, as in Figure 25. Since here $d = 1$, these edges appear in the same order around vertex $+$ of Γ_1. Recalling that loops in Γ_α correspond to edges with distinct endpoints in Γ_β, we see that the edges $A_1, A_2, \ldots, A_6$ must appear around vertex $+$ of Γ_1 as shown in Figure 27. It is easy to check that this determines the identification between all the edges of Γ_1 and Γ_2, and gives rise to the identification pattern $P(6)$ shown in Figures 28(i) and (ii).

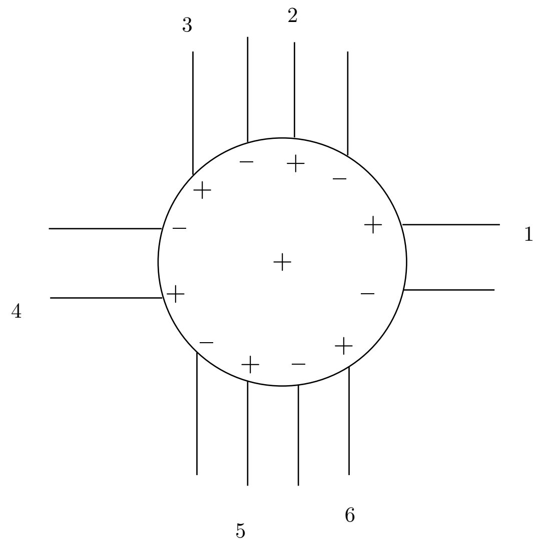


FIGURE 27

(In these figures the edges $A_1, A_2, A_3, A_4, A_5, A_6$ have been relabeled W, C, A, Y, B, and D respectively.)

In the case $\Delta = 8$ (Lemma 6.4(i)), let $A_1, A_2, \ldots, A_8$ be the edges of Γ_2 with label + at vertex + of Γ_2, numbered cyclically in such a way that A_3, A_4, A_7, and A_8 are loops (see Figure 29). There are now two possibilities for $d = d_{21}$, namely $d = 1$ and $d = 3$, giving rise to the two arrangements of $A_1, A_2, \ldots, A_8$ around vertex + of Γ_1 shown in Figures 30(i) and (ii). Again, it is straightforward to check that each case leads to a unique possibility for the identification between the edges of Γ_1 and Γ_2. The corresponding identification patterns $P(8)_1$ and $P(8)_2$ are shown in Figures 31 and 32; the graph Γ_2 in both cases is shown in Figure 31, while the graphs Γ_1 for $P(8)_1$ and $P(8)_2$ are shown in Figures 32(i) and (ii) respectively. (In these figures, $A_1, A_2, \ldots, A_8$ have been relabeled S, U, D, B, W, Y, A, C.)

We have thus shown that in Case (A), the only possibilities for F_1, F_2 are given by the identification patterns $P(6)$, $P(8)_1$ and $P(8)_2$.

7. CASE (B)

In this section we treat the case where $n_1 = 2$, the two boundary components of F_1 have opposite sign, and all the boundary components of F_2 have the same sign. Call the boundary components of F_1 (the vertices of Γ_1) + and −. Since all the boundary components of F_2 have the same sign, the parity rule implies that there are no loops in Γ_1. Hence $\Gamma_1 \cong G(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

We distinguish the four cases $n_2 = 1$, $n_2 = 2$, $n_2 = 3$, and $n_2 \geq 4$.

CASE (1). $n_2 = 1$. Here $\Gamma_2 \cong H(\beta_1, \beta_2, \beta_3)$. Note that

$$\sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{3} \beta_i = \Delta \ .$$

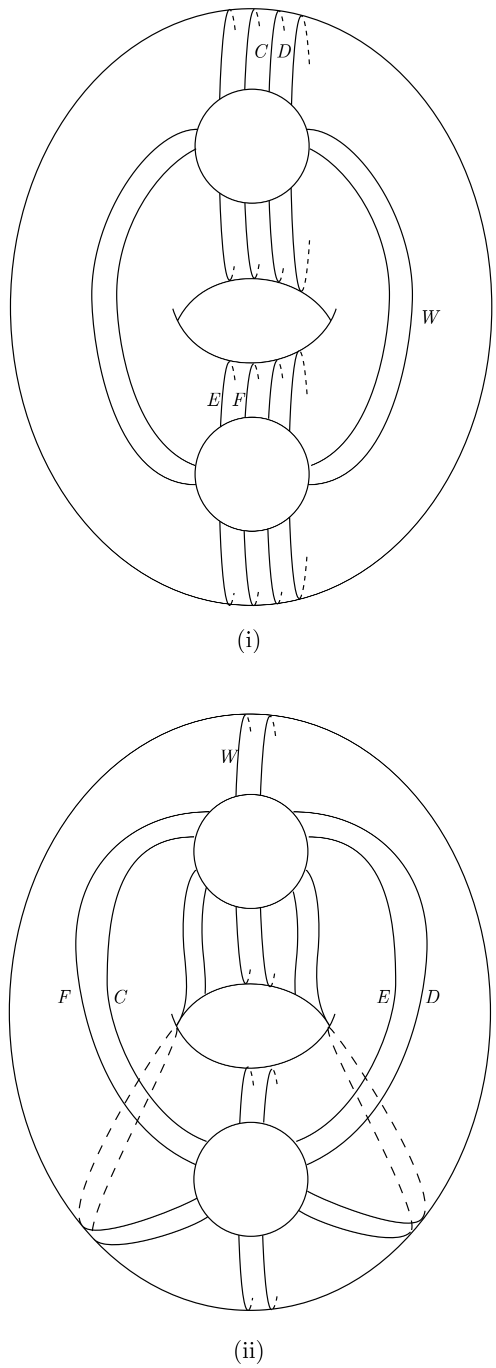


FIGURE 28

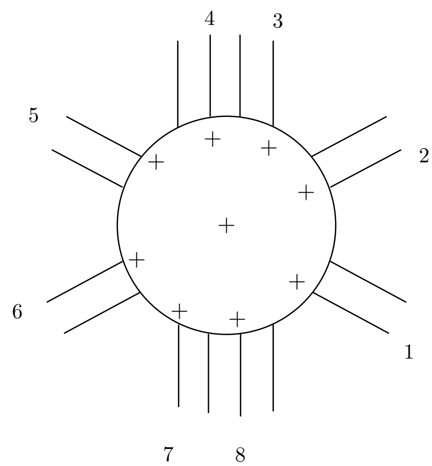


Figure 29

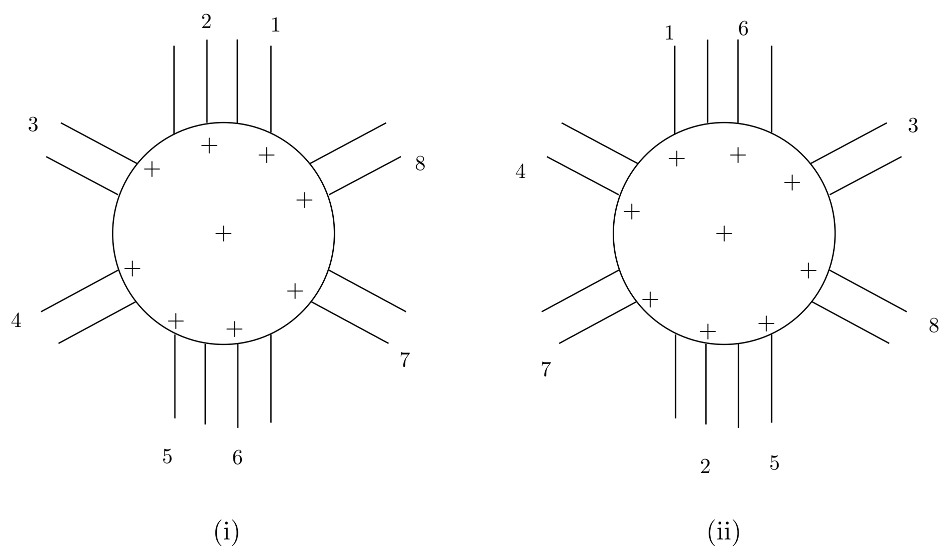


Figure 30

Lemma 7.1. (i) $\beta_i \leq 4, \quad i = 1, 2, 3$;
(ii) $\beta_1 \equiv \beta_2 \equiv \beta_3 \pmod 2$.

Proof. (i) follows from the fact that there are at most 4 parallelism classes of edges in Γ_1, together with Lemma 2.1.

Since the labels at the endpoints of a loop in any parallelism class are distinct, we see that $\beta_i \neq 0$ implies that $\beta_j + \beta_k$ is even ($\{i, j, k\} = \{1, 2, 3\}$). Since at most one β_i is zero by (i), this proves (ii). □

The only possibilities for Γ_2 allowed by Lemma 7.1 with $\Delta \geq 6$ are the following, where $d = d_{21}$:

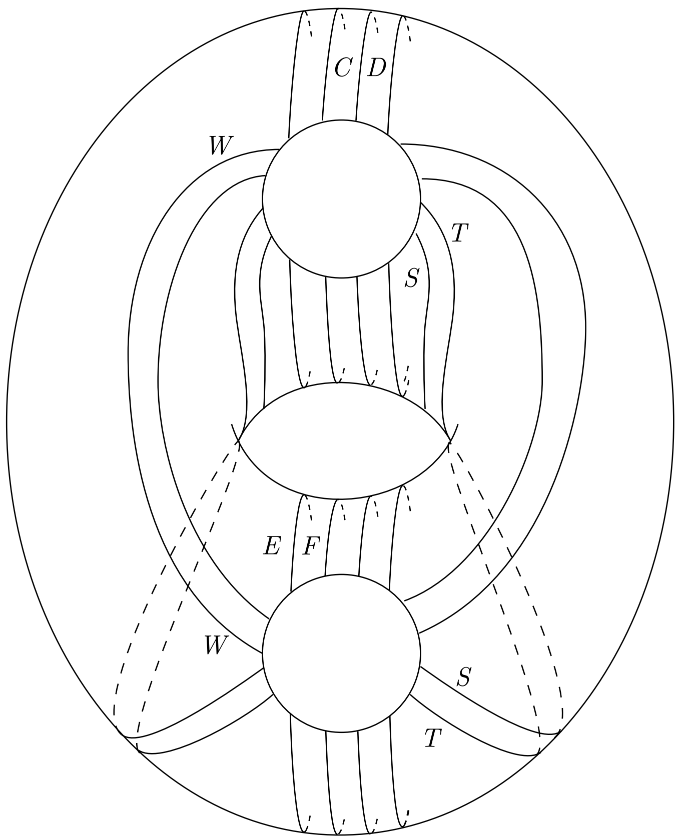


FIGURE 31

$\Delta = 12$,	$H(4,4,4)$,	$d = 1$ or 5 .
$\Delta = 10$,	$H(4,4,2)$,	$d = 1$ or 3 .
$\Delta = 9$,	$H(3,3,3)$,	$d = 1, 2$ or 4 .
$\Delta = 8$,	$H(4,4,0)$ or $H(4,2,2)$,	$d = 1$ or 3 .
$\Delta = 7$,	$H(3,3,1)$,	$d = 1, 2,$ or 3 .
$\Delta = 6$,	$H(4,2,0)$ or $H(2,2,2)$,	$d = 1$.

Consider the case $\Delta = 12$. Let the edges of Γ_2 be $A_1, A_2, \dots, A_{12}$, numbered around the vertex of Γ_2 according to the position of the end with label $+$. See Figure 33. These edges occur around the vertex $+$ of Γ_1 in the order A_1, A_{1+d}, $A_{1+2d}, \dots$. Here, $d = 1$ or 5.

First suppose that $d = 1$. Then (interpreting subscripts modulo 12) no consecutive pair A_i, A_{i+1} can be parallel in Γ_1, either because they are parallel in Γ_2 (and by Lemma 2.1), or because their endpoints violate Lemma 2.5(i) (or, equivalently here, Lemma 2.5(iii)). But this contradicts the form of Γ_1. Similarly, in the case $d = 5$, we see that no pair A_i, A_{i+5} can be parallel in Γ_1.

The other cases are handled in the same way, by examining the arrangement of the edges $A_1, \dots, A_\Delta$ around the vertex of Γ_2. For instance, in the case $\Delta = 10$,

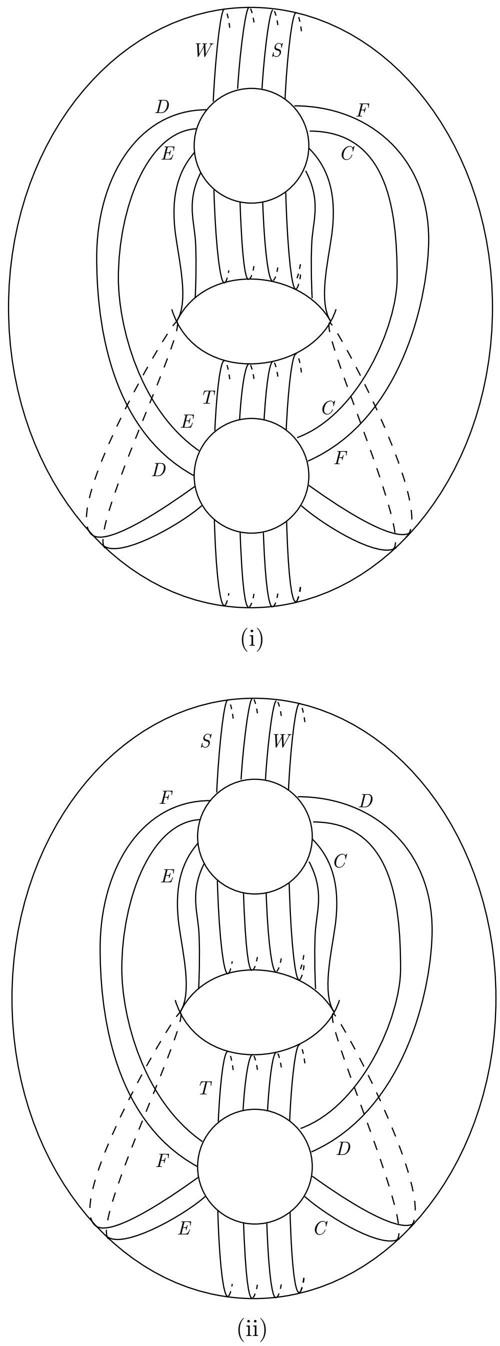


Figure 32

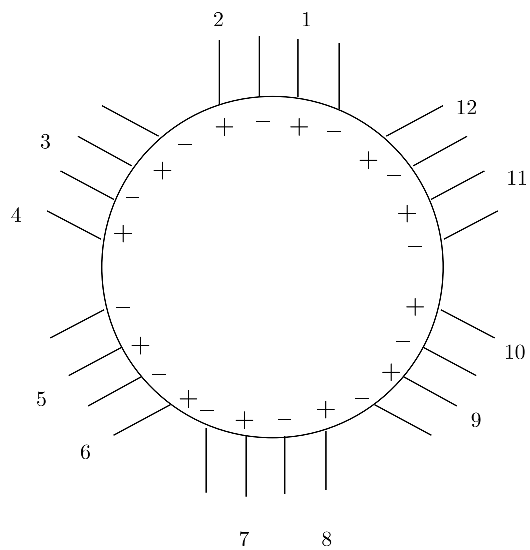


FIGURE 33

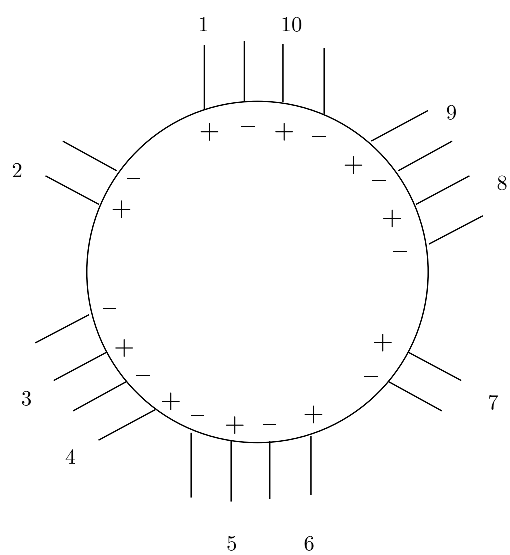


FIGURE 34

$d = 3$, we see using Lemma 2.5(i) that the only pairs A_i, A_{i+3} that can be parallel in Γ_1 are A_1 and A_4, A_{10} and A_3, A_6 and A_9, and A_5 and A_8. See Figure 34. Since Γ_1 has at most 4 parallelism classes of edges, this is a contradiction.

One readily checks that the only possibilities not ruled out by this means are $\Delta = 7$, $d = 2$, and $\Delta = 6$, $\Gamma_2 \cong H(2,2,2)$. These do in fact occur, with $\Gamma_1 \cong G(2,1,2,2)$ and $G(3,0,0,3)$ respectively. The (unique) corresponding identification patterns of the edges of Γ_1 and Γ_2, $P(7)$ and $P(6)_2$, are shown in Figures 35 and

36 respectively. We shall see later, however, in Section 11, that the case $P(6)_2$ is topologically degenerate.

CASE (2). $n_2 = 2$. Here $\Gamma_2 \cong G(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$. Call the boundary components of F_2 (the vertices of Γ_2) x and y.

Lemma 7.2. $\Delta \leq 5$.

Proof. Note that $\sum_{i=1}^4 \alpha_i = 2\Delta$. Also, since $n_1 = n_2 = 2$, Lemma 5.3 applies. In particular, $\alpha_i \leq 4$, $i = 1, 2, 3, 4$. Hence $\Delta \leq 8$. It remains to consider the cases $\Delta = 8$, $\Delta = 7$, and $\Delta = 6$.

$\Delta = 8$ *is impossible*. Here $\sum_{i=1}^4 \alpha_i = 16$. Hence, by Lemma 5.3, $\alpha_i = 4$ and $\varepsilon_i = 1$, $i = 1, 2, 3, 4$. In particular, there are no loops in Γ_2. Therefore $\Gamma_1 \cong \Gamma_2 \cong G(4, 4, 4, 4)$.

The arrangement of the endpoints of the edges of Γ_1 at the vertices + and − is as shown in Figure 37. Let the edges with label x at vertex + be $A_1, A_2, \ldots, A_8$ as indicated. In Γ_2, these are the edges with label + at x and label − at y. Around the vertex x, they appear in the order $A_1, A_{1+d} A_{1+2d}, \ldots$, where $d = d_{12}$. Here the two possibilities for d are 1 and 3.

Consider first the case $d = 1$. Then no pair A_i, A_{i+1} (subscripts understood modulo 8) can be parallel in Γ_2, either because they are parallel in Γ_1, or because their endpoints at the vertices + and − of Γ_1 violate Lemma 2.5(i). This is a contradiction. Similarly, if $d = 3$, one easily checks that no pair A_i, A_{i+3} can be parallel in Γ_2 because of Lemma 2.5(i).

$\Delta = 7$ *is impossible*. Here $\sum_{i=1}^4 \alpha_i = 14$. From Lemma 5.3 it easily follows that $\Gamma_1 \cong G(4, 4, 4, 2)$, and $\varepsilon_i = 1$, $i = 1, 2, 3, 4$. Hence there are no loops in Γ_2, so $\Gamma_2 \cong G(\beta_1, \beta_2, \beta_3, \beta_4)$, say.

By Lemma 2.1, the edges in any parallelism class in Γ_1 must belong to distinct parallelism classes in Γ_2. Therefore $(\beta_1, \beta_2, \beta_3, \beta_4)$ is some permutation of $(4, 4, 3, 3)$. But this contradicts Lemma 5.3.

$\Delta = 6$ *is impossible*. Here $\sum_{i=1}^4 \alpha_i = 12$. It is straightforward to check that the only possibilities for Γ_1 allowed by Lemma 5.3 are: $G(4, 4, 4, 0)$, $G(4, 4, 2, 2)$, $G(4, 2, 4, 2)$, and $G(3, 3, 3, 3)$. Since all ε_i's are 1 in all cases, there are no loops in Γ_2. So $\Gamma_2 \cong G(\beta_1, \beta_2, \beta_3, \beta_4)$.

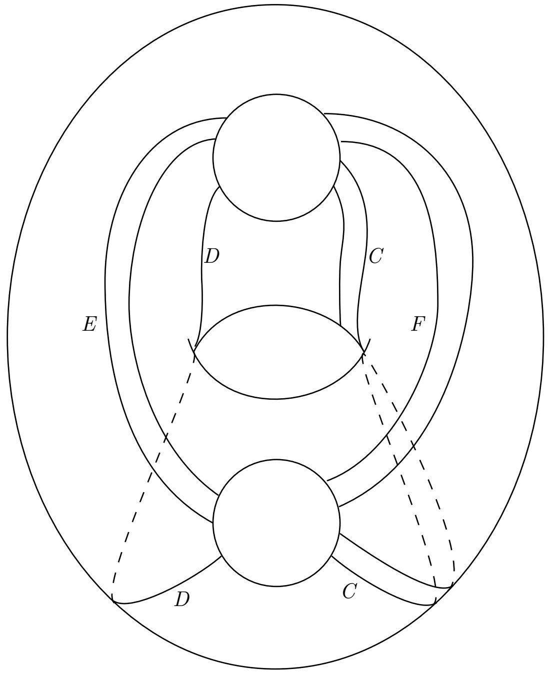
Let $A_1, A_2, \ldots, A_6$ be the edges of Γ_1 with label x at vertex +, numbered in order around that vertex. Since here $d = 1$, these edges appear in the same order around vertex x of Γ_2. We consider the four possibilities listed above for Γ_1.

Suppose $\Gamma_1 \cong G(4, 4, 4, 0)$. See Figure 38. We then see that no consecutive pair A_i, A_{i+1} can be parallel in Γ_2, either because they are parallel in Γ_1 or because of Lemma 2.5(i). This contradicts the form of Γ_2.

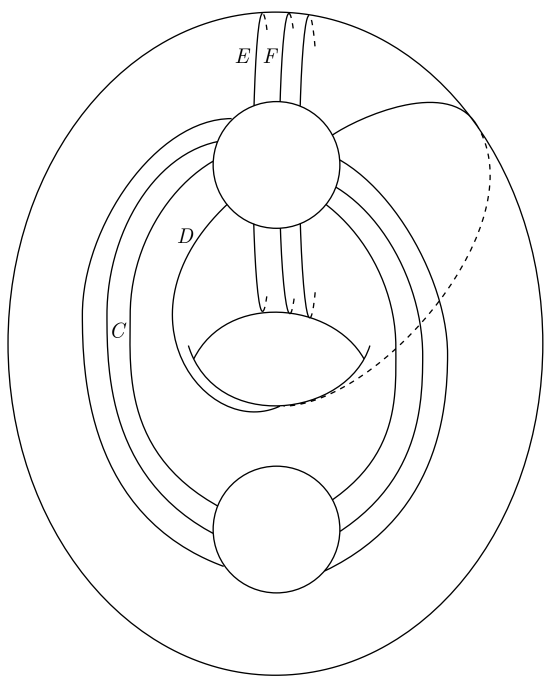
Suppose $\Gamma_1 \cong G(4, 4, 2, 2)$. See Figure 39. Here, the only consecutive pair A_i, A_{i+1} that is permitted to be parallel in Γ_2 by the two restrictions used above is A_5, A_6. But this would lead to too many parallelism classes on Γ_2.

In the other two cases, $G(4, 2, 4, 2)$ and $G(3, 3, 3, 3)$, we argue similarly and easily conclude that no pair A_i, A_{i+1} can be parallel in Γ_2. □

We now consider the case $n_2 \geq 3$. Let P_1, P_2, P_3, P_4 be the parallelism classes of edges in Γ_1, where P_i has size α_i. Let P_i^+, P_i^- denote the set of labels at the end of P_i at vertex +, − respectively. If an edge in P_i has label x at vertex +, then it has label $\pi_i(x) \equiv x + p_i \pmod{n_2}$ at − for some p_i. Thus we have a

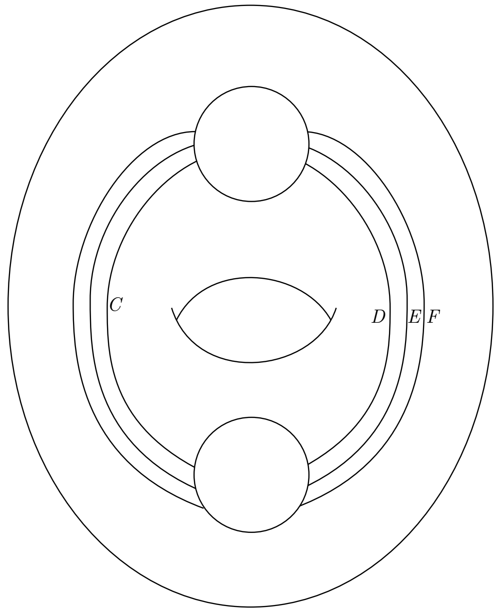


(i)

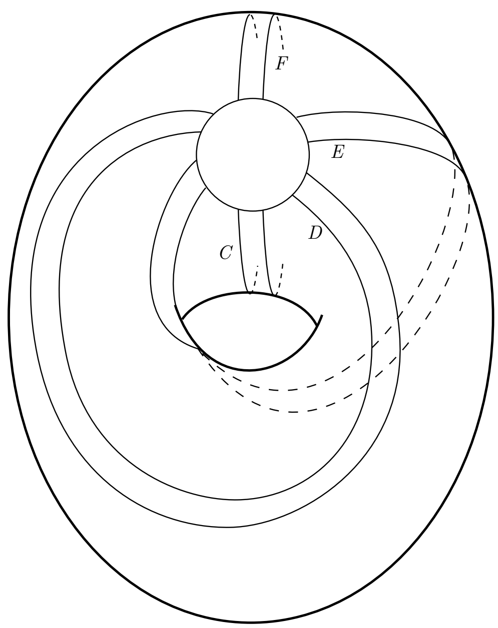


(ii)

FIGURE 35



(i)



(ii)

FIGURE 36

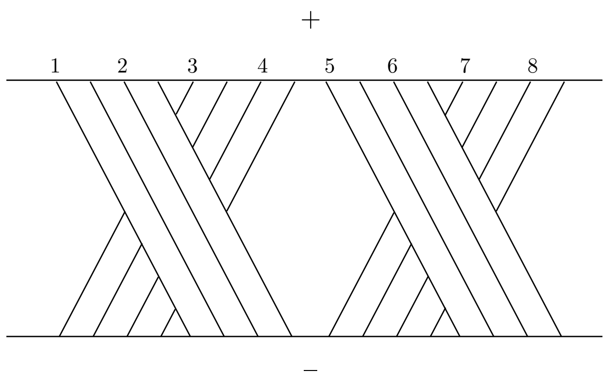


FIGURE 37

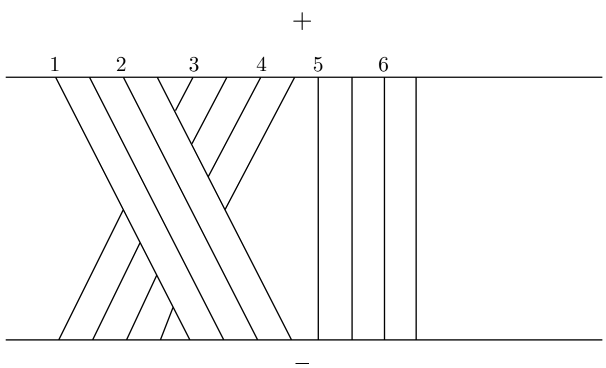


FIGURE 38

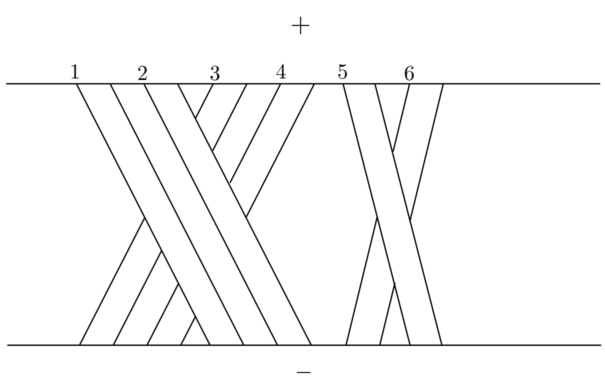


FIGURE 39

well-defined permutation π_i of $\{1, 2, \ldots, n_2\}$, even if $\alpha_i < n_2$ (but provided that $\alpha_i \neq 0$), $i = 1, 2, 3, 4$.

Note that since $\sum \alpha_i = \Delta n_2 \geq 6n_2$, some $\alpha_i \geq n_2 + 1$, and hence the corresponding π_i has only one orbit, by Lemma 4.2.

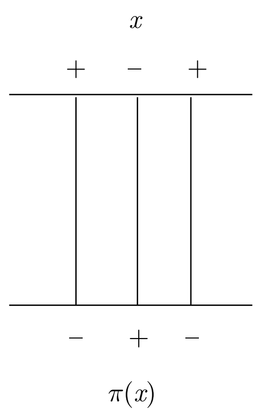


FIGURE 40

Lemma 7.3. *The permutations π_i, $i = 1, 2, 3, 4$, are not all equal.*

Proof. Suppose $\pi_i = \pi$, $i = 1, 2, 3, 4$. Then for any vertex x of Γ_2, there are Δ edges in Γ_1 with label x at vertex $+$ and label $\pi(x)$ at vertex $-$. Since $\nu(x, \pi(x)) \leq 5$ by Lemma 5.4(i), some pair of these edges must be parallel in Γ_2. This then gives rise to an edge in Γ_1 with label x at $-$ and label $\pi(x)$ at $+$ (see Figure 40). But this implies that π^2 is the identity, and hence $n_2 = 2$ (since π has only one orbit), a contradiction. □

We now distinguish the two cases $n_2 = 3$ and $n_2 \geq 4$.

CASE (3). $n_2 = 3$. In the first two lemmas we use the convention that $\{x, y, z\} = \{1, 2, 3\}$, the vertices of Γ_2.

Lemma 7.4. *(i) If Γ_2 contains a loop at x, then $\nu(y, z) \leq 2$;*
(ii) If $\nu(x, y) \geq 4$, then $\nu(x, z) \leq 2$.

Proof. (i) Cutting the torus $\widehat{F}_2$ along a loop at x gives an annulus containing y and z in its interior. The result is now clear.

(ii) If $\nu(x, y) \geq 2$, cutting $\widehat{F}_2$ along two edges of $\overline{\Gamma}_2$ joining x and y gives an annulus with z in its interior. If there are two additional edges in $\overline{\Gamma}_2$ joining x and y, then they must be as shown in Figure 41(i), (ii) or (iii). The fact that $\nu(x, z) \leq 2$ now follows by inspection. □

Lemma 7.5. *Suppose that $\nu(x, y) = \nu(x, z) = 3$. Let Y_1, Y_2, Y_3 and Z_1, Z_2, Z_3 be the edges of $\overline{\Gamma}_2$ joining x to y and z respectively. Then (for suitable choice of subscripts) around the vertex x these edges appear in the order $Y_1, Z_1, Y_2, Z_2, Y_3, Z_3$.*

Proof. Consider three consecutive parallel edges of Γ_1 in some family of at least four parallel edges. Cutting the torus $\widehat{F}_2$ along the corresponding edges of $\overline{\Gamma}_2$ gives an annulus N.

Since $\nu(x, y) = \nu(x, z) = 3$ by hypothesis, there are additional edges Y, Y' and Z, Z' of $\overline{\Gamma}_2$ joining x to y and z respectively. Such an edge is of either type I or type II (see the proof of Lemma 5.4).

If both Y and Y' are of type I, then $\nu(x, z) = 1$ (see Figure 42).

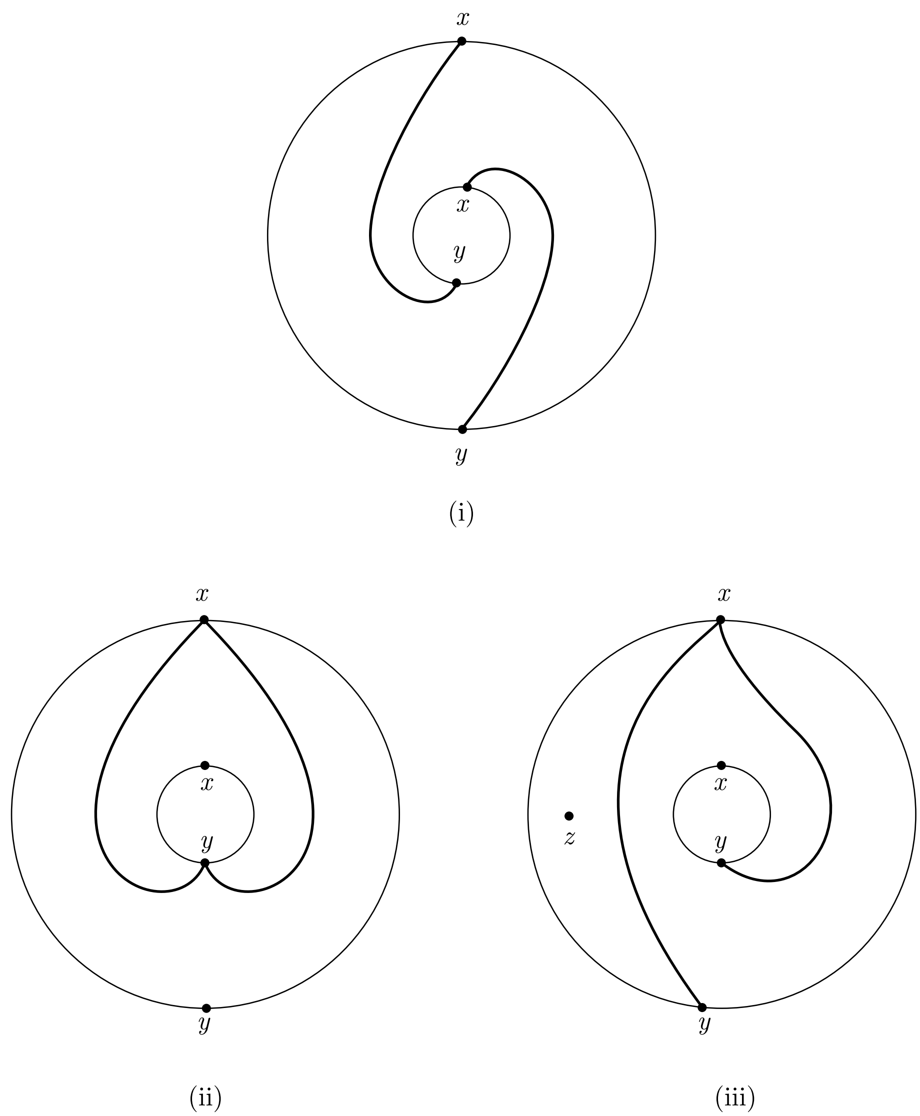


Figure 41

If Y is of type I and Y' of type II, then we must have the situation shown in Figure 43.

If both Y and Y' are of type II, then we must have the situation shown in either Figure 44(i) or Figure 44(ii).

The result follows by inspection. □

Let σ denote the permutation (123). Then $\pi_i = \sigma$, σ^{-1}, or id.

Lemma 7.6. *(i) If $\pi_i = id$ then $\alpha_i \leq 3$.*
(ii) $\alpha_i \leq 9$, $i = 1, 2, 3, 4$.

Proof. (i) follows from Lemma 4.2.

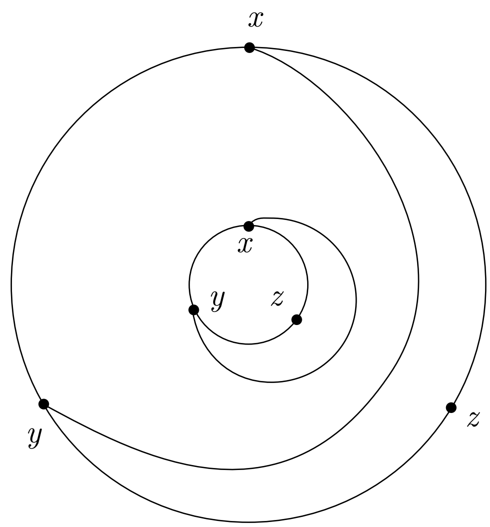


FIGURE 42

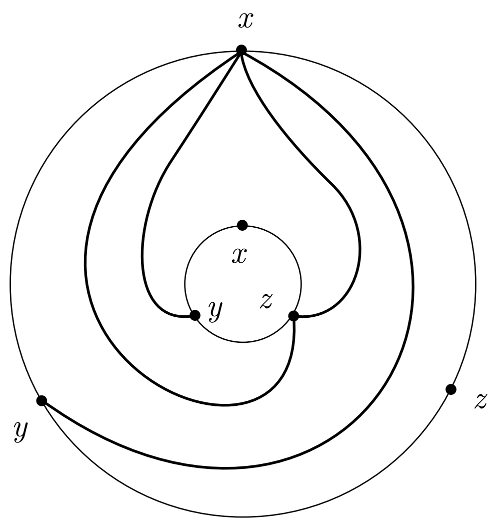


FIGURE 43

(ii) Suppose $\pi_i \neq id$ but $\alpha_i > 9$. Then for some x, y we have $\nu(x, y) \geq 4$ and $\nu(x, z) \geq 3$ (and $\nu(y, z) \geq 3$), by Lemma 2.1. This contradicts Lemma 7.4(ii). □

As a preliminary restriction on the possibilities for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ we consider the residue classes of the α_i's modulo 3. Without loss of generality, we assume that 1 is the first label in P_1^+ as we go round the vertex + anticlockwise.

Lemma 7.7. *The only possibilities (up to cyclic permutation and reversal) for* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *and the corresponding permutations* $(\pi_1, \pi_2, \pi_3, \pi_4)$ *are:*
(a) $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv (1, 2, 0, 0) \pmod 3$, $(\pi_1, \pi_2, \pi_3, \pi_4) = (\sigma, \sigma, \sigma^{-1}, \sigma^{-1})$;
(b) $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv (1, 1, 2, 2) \pmod 3$, $(\pi_1, \pi_2, \pi_3, \pi_4) = (\sigma, \sigma^{-1}, \sigma^{-1}, \sigma)$;

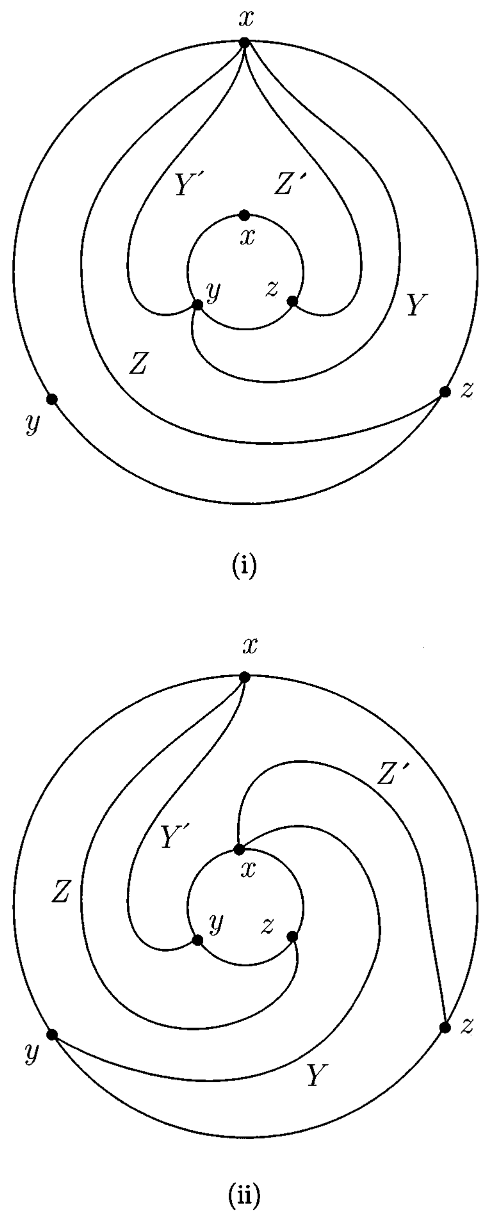


FIGURE 44

(c) $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv (1, 0, 2, 0) \pmod 3$, $(\pi_1, \pi_2, \pi_3, \pi_4) = (\sigma, id, \sigma, \sigma^{-1})$.

Proof. Since $\sum \alpha_i = 3\Delta$, the only possibilities for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ modulo 3 are: (0,0,0,0), (1,1,1,0), (2,2,2,0), (1,2,0,0), (1,0,2,0), (1,1,2,2) and (1,2,1,2). We examine each of these in turn. Note also that $\sum \alpha_i \geq 18$.

(0,0,0,0). Here $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3.

(1,1,1,0). Going around the vertex $-$ clockwise, the first label in P_2^- is either 1, 2, or 3. This gives three possibilities for the labeling at $-$ modulo 3, illustrated in

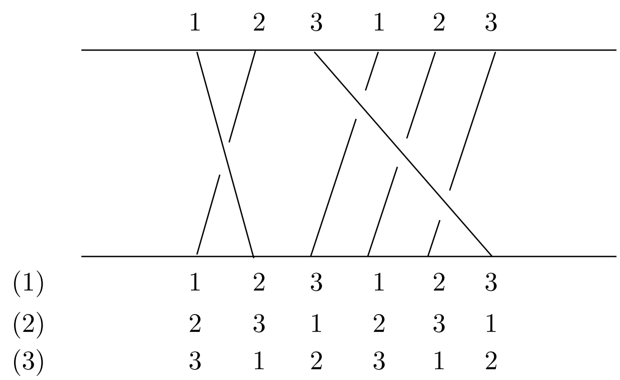


FIGURE 45

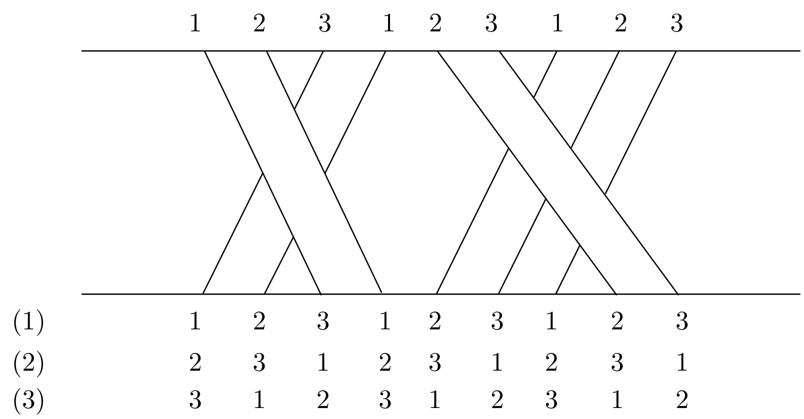


FIGURE 46

Figure 45 (1), (2), and (3) (α_4 may be 0). (Here, and subsequently, we make the convention that the vertex $+$ is at the top.)

(1) By Lemma 7.6(i), $\alpha_3 = 1$. Also $\nu(1,2) \leq 2$ by Lemma 7.4(i). Therefore, by Lemma 2.1, there are at most two edges in P_i with label 1 at one endpoint and label 2 at the other, $i = 1, 2, 4$. Thus $\alpha_1, \alpha_2 \leq 6$ and $\alpha_4 \leq 7$. By considering the residues of α_1, α_2, and α_4 modulo 3 this gives $\alpha_1, \alpha_2 \leq 4$ and $\alpha_4 \leq 6$. Hence $\sum \alpha_i \leq 4 + 4 + 1 + 7 = 15$, a contradiction.

(2) Here $\alpha_2 = 1$ by Lemma 7.6(i), and $\nu(1,3) \leq 2$ by Lemma 7.4(i). Therefore $\alpha_1, \alpha_3 \leq 6$ (and hence ≤ 4). Also $\alpha_4 \leq 3$ by Lemma 7.6(i). Hence $\sum \alpha_i \leq 12$, a contradiction.

(3) This case is isomorphic to case (1) above.

(2,2,2,0). Again we have three possible labelings modulo 3 at the vertex $-$, illustrated in Figure 46 (1), (2) and (3).

(1) Here $\alpha_3 = 2$, and $\nu(1,3) \leq 2$, $\nu(1,2) \leq 2$. Hence $\alpha_1, \alpha_2 \leq 5$ and $\alpha_4 \leq 6$. Since $\sum \alpha_i \geq 18$, the only possibility is $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (5, 5, 2, 6)$. Let A, B, C be the edges of Γ_1 shown in Figure 47. By Lemma 2.1, A and B are not parallel in Γ_2, and hence, since $\nu(1,3) \leq 2$, C must be parallel in Γ_2 to either A or B. But this is impossible by Lemma 2.5(i).

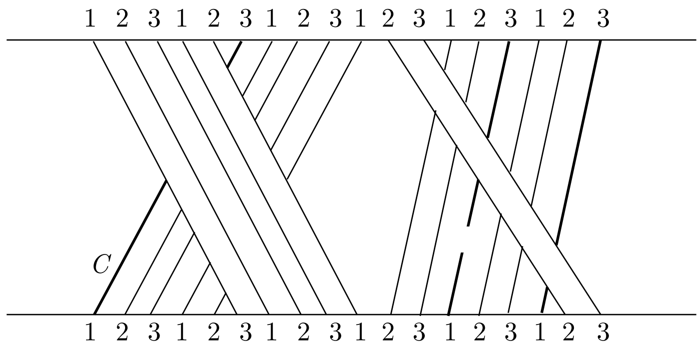


FIGURE 47

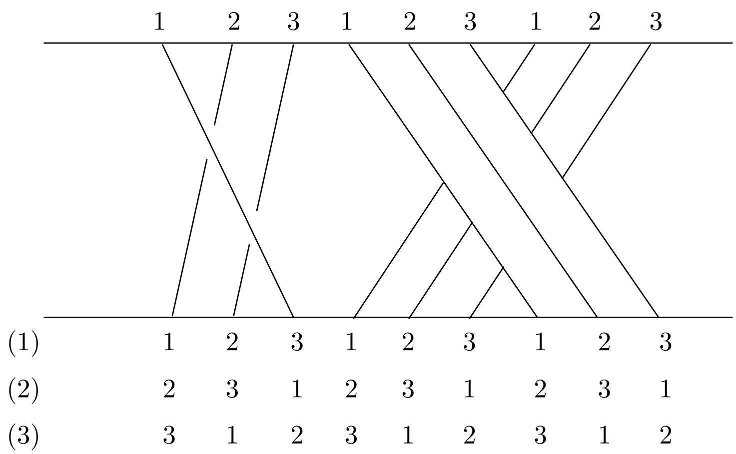


FIGURE 48

(2) This case in isomorphic to (1).

(3) Here $\alpha_3 = 2$. Hence $\alpha_1, \alpha_3 \leq 6$, and so ≤ 5. Also $\alpha_4 \leq 3$. Therefore $\sum \alpha_i \leq 15$, a contradiction.

(1,2,0,0). The three possible labelings modulo 3 at the vertex $-$ are illustrated in Figure 48 (1), (2), and (3).

(1) Here α_3 and α_4 are either 0 or 3. If both are 0, then $\alpha_1 + \alpha_2 \geq 18$, and hence α_1 or $\alpha_2 > 9$, contradicting Lemma 7.6(ii). If either α_3 or α_4 is 3, then $\nu(x, y) \leq 2$ for all x, y, giving $\alpha_1 \leq 4$ and $\alpha_2 \leq 5$. Hence $\sum \alpha_i \leq 15$.

(2) Here $\alpha_1 = 1$ and $\alpha_2 = 2$, and $\nu(x, y) \leq 2$ for all x, y. Hence $\alpha_3, \alpha_4 \leq 6$. Therefore $\sum \alpha_i \leq 15$.

(3) is possibility (a) of the lemma.

(1,0,2,0). The three possible labelings modulo 3 at the vertex $-$ are illustrated in Figure 49 (1), (2), and (3).

(1) Here $\alpha_1 = 1$, $\alpha_3 = 2$, and hence $\alpha_2, \alpha_4 \leq 6$, giving $\sum \alpha_i \leq 15$.

(2) This is possibility (c) of the lemma.

(3) is isomorphic to (2).

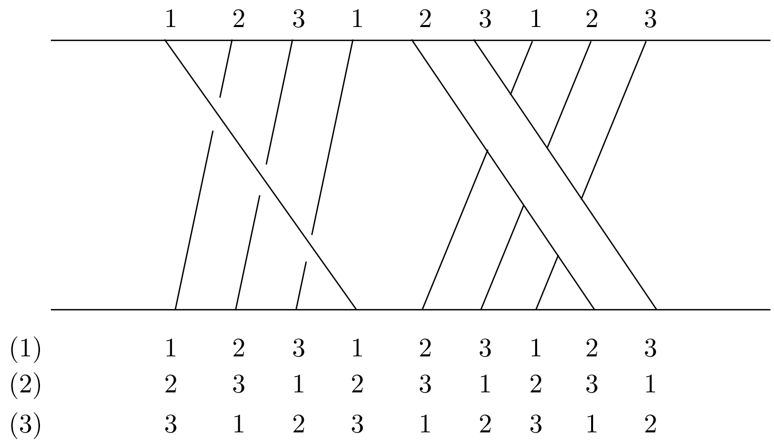


FIGURE 49

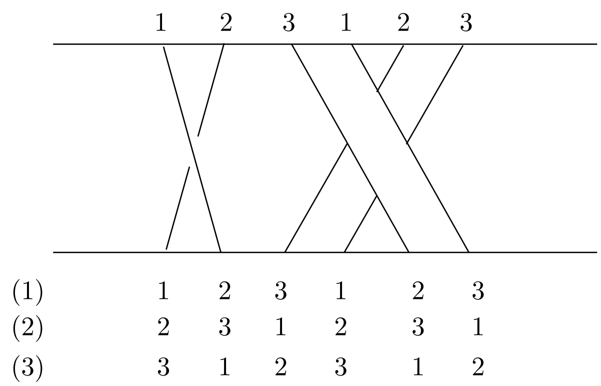


FIGURE 50

(1,1,2,2). The three possible labelings at $-$ are illustrated in Figure 50 (1), (2), and (3).

(1) is possibility (b) of the lemma.

(2) and (3) are ruled out by arguments similar to those given above.

(1,2,1,2). Here $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3. □

We now proceed to rule out the possibilities left after Lemma 7.7. Since $\alpha_1 \leq 9$ by Lemma 7.6(ii), there are only a finite number of cases to consider.

Lemma 7.8. *Case (a) of Lemma 7.7 is impossible.*

Proof. Since $\alpha_i \leq 9$, we have that $\alpha_1 = 1, 4$, or 7, $\alpha_2 = 2, 5$, or 8, and $\alpha_3, \alpha_4 = 0, 3, 6$, or 9.

As an example, suppose that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (*, *, 6, 9)$. Then the edges of Γ_1 in $P_3 \cup P_4$ are as in Figure 51. Since P_4 contains three edges with label 1 at $+$ and 3 at $-$, $\nu(1,3) \geq 3$ by Lemma 2.1. Now let A, B, C be the three edges in P_4 with label 2 at $+$ and 1 at $-$, and let X be one of the edges in P_3 with label 2 at $+$ and 1 at $-$. Let $A^{\pm}$ denote the endpoints of A at $\pm$, and similarly for B, C, and X. One

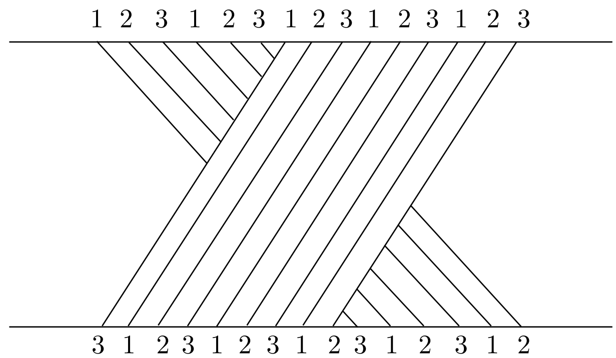


FIGURE 51

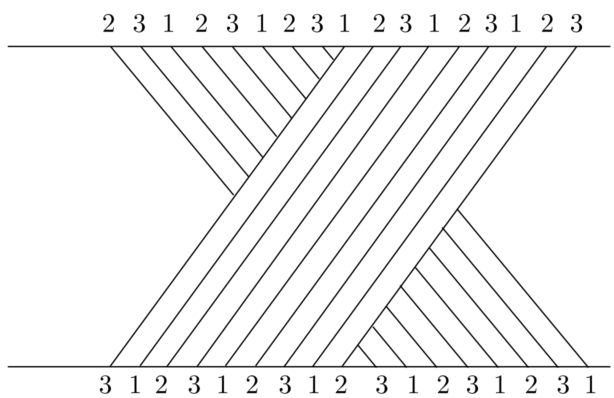


FIGURE 52

readily checks that $\delta_1(X^+, Y^+) \neq \delta_1(X^-, Y^-)$ for $Y = A, B$, or C, and hence X is not parallel in Γ_2 to either A, B, or C, by Lemma 2.5(i). Since A, B and C are pairwise non-parallel in Γ_2 by Lemma 2.1, we get $\nu(1,2) \geq 4$. But this contradicts Lemma 7.4(ii).

The case $(*, *, 9, 6)$ is ruled out in exactly the same way, and a similar argument rules out $(7, 8, *, *)$.

Now consider the case $(*, 8, 9, *)$, illustrated in Figure 52. Again, considering P_3 shows that $\nu(x, y) \geq 3$ for all x, y, by Lemma 2.1. Let A, B, C be the edges in P_3 with label 1 at $+$ and 3 at $-$, and let X be one of the edges in P_2 with label 3 at $+$ and 1 at $-$. One readily verifies, this time using Lemma 2.5(ii), that X is not parallel in Γ_2 to either A, B, or C. Hence $\nu(1,3) \geq 4$, contradicting Lemma 7.4(ii) as before.

Similar arguments rule out the cases $(*, 2, 9, *)$, $(*, 8, 3, *)$, and $(4, *, *, 9)$.

Next consider the case $(*, 5, 6, *)$, shown in Figure 53. By considering the four edges in $P_2 \cup P_3$ with label 2 at one end and 3 at the other, and using Lemmas 2.1 and 2.5(iii), we see that $\nu(2,3) \geq 4$. Similarly, $\nu(3,1) \geq 4$. But this contradicts Lemma 7.4(ii).

A similar argument rules out $(7, *, *, 6)$.

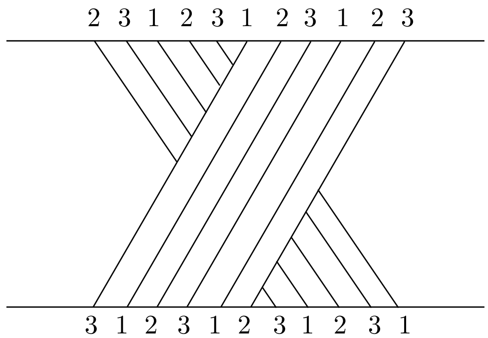


FIGURE 53

Since $\alpha_3 \equiv 0 \pmod 3$, the argument used to rule out $(*, 8, 9, *)$ applies essentially verbatim to rule out $(*, 8, *, 9)$; one just replaces P_3 by P_4. In the same way, the arguments in the next five cases also rule out, respectively, $(*, 2, *, 9)$, $(*, 8, *, 3)$, $(4, *, 9, *)$, $(*, 5, *, 6)$, and $(7, *, 6, *)$,

To eliminate some more possibilities, we use a different line of argument. Since Γ_1 has at most four parallelism classes of edges, the number of edges in any parallelism class in Γ_2 is at most four, by Lemma 2.1. Hence if two parallel edges in Γ_2 have label $+$ (say) at a vertex x, then their endpoints at x are adjacent among all edge endpoints with label $+$ at x. Therefore the endpoints at the vertex $+$ of the corresponding edges in Γ_1, which have label x, are "d apart" among all edge endpoints with label x at $+$ (where $d = d_{21}$), i.e., are separated by $d - 1$ such endpoints.

To see how this may be used, consider the case $(4, 8, *, *)$, illustrated in Figure 54. Since P_2 contains three edges with endpoint labels 1 and 3, $\nu(1, 3) \geq 3$. Therefore $\nu(2, 3) \leq 3$ by Lemma 7.4(ii). Let A, B, C be the edges in P_2 with label 2 at $+$ and 3 at $-$. (We adopt the convention that if we label edges in a given P_i alphabetically, then the alphabetic ordering agrees with the anticlockwise ordering around the vertex $+$; see edges A and B in Figure 47.) Let X be the edge in P_1 with label 2 at $+$ and 3 at $-$. Since $\nu(2, 3) \leq 3$, X must be parallel in Γ_2 to either A, B, or C. By Lemma 2.5(i), Xmust be parallel to B. Therefore $d = 2$, and hence Δ is odd. This rules out $(4, 8, 6, 0)$ $(= (4, 8, 0, 6))$, and (4,8,6,6), which have $\Delta = 6$ and 8 respectively.

Similarly, one shows that for $(7, 5, *, *)$, $(*, *, 6, 6)$, $(*, *, 9, 3)$, and $(*, *, 3, 9)$, we must have $d = 2$, and hence Δ odd. This rules out (1,5,9,3), (1,5,3,9), (4,2,6,6), (7,5,3,3), (7,5,3,9), (7,5,9,3), and (7,5,9,9).

The only cases left are (1,5,9,9), (7,5,0,9), and (1,8,6,6).

Consider (1,5,9,9). The argument just given (applied to P_3 and P_4) shows that for $(*, *, 9, 9)$ we must have $d = 3$. Now let A, B, C be the edges in P_3 with label 2 at $+$ and label 1 at $-$, and let X, Y be the edges in P_1, P_2 respectively with label 1 at $+$ and label 2 at $-$. See Figure 55. Since $\nu(1, 2) \leq 3$, X must be parallel in Γ_2 to A, B, or C, and similarly for Y. Using Lemma 2.5(ii), we see that X and Y must be parallel to B, and hence to each other. But this implies $d = 1$.

Finally, the cases (7,5,0,9) and (1,8,6,6) will be eliminated using Lemma 7.5.

Consider (7,5,0,9). Let A, B, C be the edges in P_4 with label 1 at $+$ and 3 at $-$. Let X, Y and U, V be the edges in P_1 and P_2 respectively with label 3 at $+$ and 1

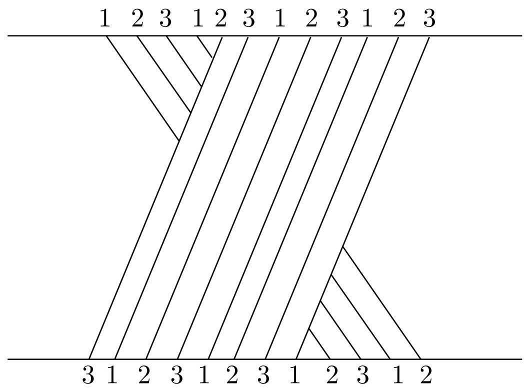


FIGURE 54

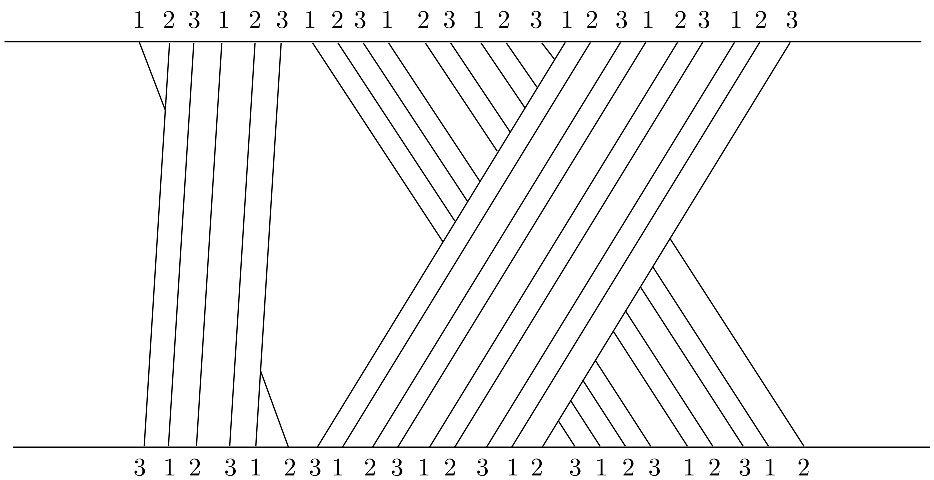


FIGURE 55

at $-$ (see Figure 57). Using Lemmas 2.1 and 2.5(ii), and the fact that $\nu(1,3) \leq 3$, we see that these seven edges fall into exactly the following parallelism classes in $\Gamma_2 : \{A\}, \{B, X, U\}, \{C, Y, V\}$. Since X is parallel to U, we have $d = 2$. Hence A and C are adjacent at vertex 1 among edges of Γ_2 with label $+$ at 1. The two parallelism classes $\{A\}$ and $\{C, Y, V\}$ then give rise to two edges of $\overline{\Gamma}_2$ that join 1 to 3 and are adjacent at 1. Since $\nu(1,3) = \nu(1,2) = 3$, this contradicts Lemma 7.5.

(1,8,6,6) can be handled similarly, by considering the three edges in P_2 with label 3 at $+$, 1 at $-$, and the four edges in $P_3 \cup P_4$ with label 1 at $+$, 3 at $-$. See Figure 56. □

Lemma 7.9. *Case (b) of Lemma 7.7 is impossible.*

Proof. Here $\alpha_1, \alpha_2 = 1, 4$, or 7, and $\alpha_3, \alpha_4 = 2, 5$, or 8.

Exactly as in the proof of the previous lemma, $(*, 7, 8, *)$, $(7, *, *, 8)$, $(*, *, 5, 8)$, and $(*, *, 8, 5)$ are impossible, and in a similar way one rules out $(4, 7, *, *)$ and $(7, 4, *, *)$.

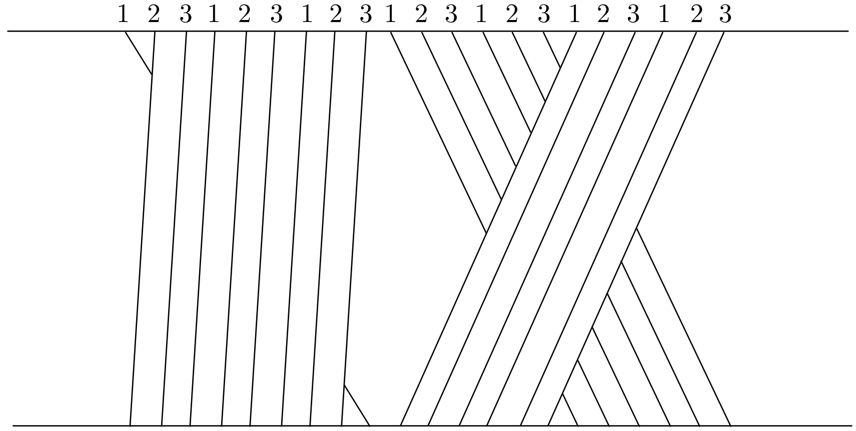


FIGURE 56

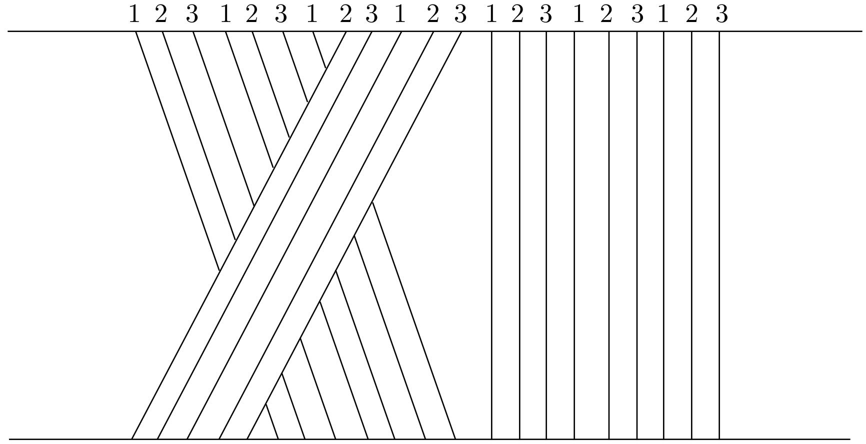


FIGURE 57

Also, $(*, 7, *, 8)$ is ruled out by considering the three edges A, B, C in P_4 with label 2 at + and 3 at −, and an edge X in P_2 with label 3 at + and 2 at −. See Figure 58. One checks that by Lemmas 2.1 and 2.5(ii), X, A, B, C are pairwise non-parallel in Γ_2, contradicting $\nu(2, 3) \leq 3$. Similarly, $(7, *, 8, *)$ is impossible.

Again as in the proof of the previous lemma, in the cases $(*, 7, 5, *)$, $(7, *, *, 5)$, $(*, 4, 8, *)$, and $(4, *, *, 8)$ we must have $d = 2$, and so Δ is odd.

It is straightforward to check that the only cases left after these exclusions are (1,1,8,8), (4,4,5,5), (1,4,8,8), (7,7,2,2), and (7,7,5,2).

First consider (1,1,8,8). Let A, B, C be the edges in P_3 with label 3 at + and 2 at −, and X, Y, Z the edges in P_4 with label 2 at + and 3 at − (see Figure 59). By arguments that are by now familiar, these edges fall into the parallelism classes $\{A, X\}$, $\{B, Y\}$, and $\{C, Z\}$ in Γ_2. Here $\Delta = 6$, and so $d = 1$. Therefore A and B, and B and C, are adjacent at vertex 3 of Γ_2 among edges with label + at 3.

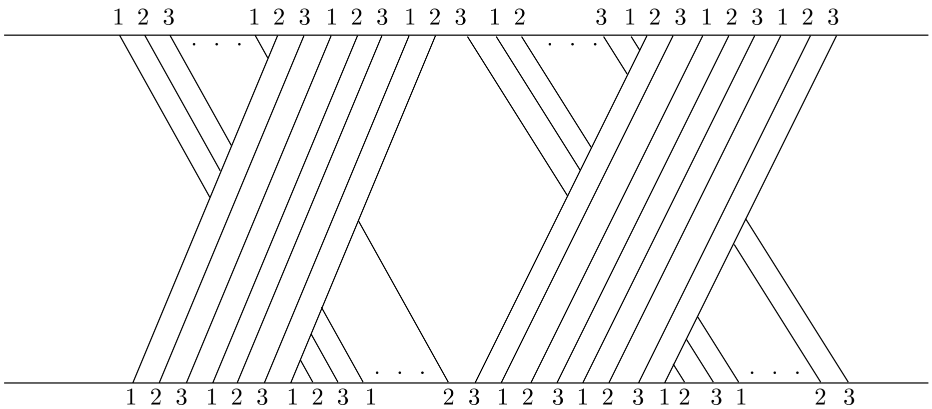


FIGURE 58

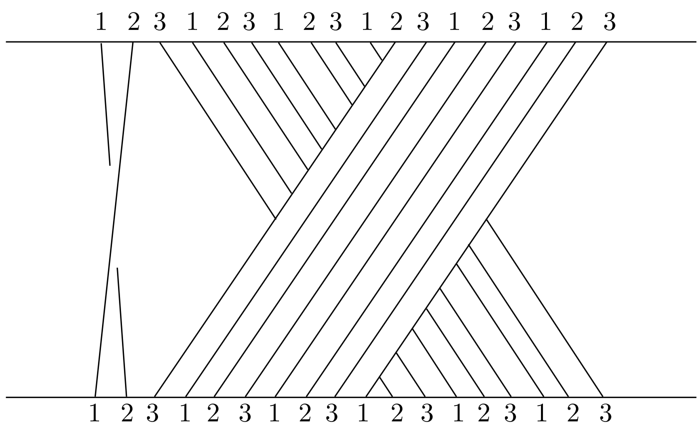


FIGURE 59

Similarly X and Y, and Y and Z, are adjacent at 3 among edges with label $-$ at 3. Thus $\{A, X\}$, $\{B, Y\}$, and $\{C, Z\}$ give rise to three edges in $\overline{\Gamma}_2$ joining vertices 3 and 2 that are adjacent at 3. This contradicts Lemma 7.5.

The case (1,4,8,8) is handled similarly, using the fact that here $d = 2$. Specifically, if A, B, C, X, Y, Z are the edges described above, then A and C, and X and Z, are adjacent at vertex 3 among edges with label $+$ (resp. $-$) at 3. Therefore $\{A, X\}$ and $\{C, Z\}$ give rise to edges in $\overline{\Gamma}_2$ joining 3 to 2 that are adjacent at 3, again contradicting Lemma 7.5.

The remaining cases, (4,4,5,5), (7,7,2,2), and (7,7,5,2), are ruled out by arguments analogous to those just given, using Lemma 7.5. Note that $d = 1$ in the first two cases and $d = 2$ in the third. We omit the details. □

Lemma 7.10. *Case (c) of Lemma 7.7 is impossible.*

Proof. Here $\alpha_2 = 0$ or 3.

If $\alpha_2 = 3$ then $\nu(x, y) \leq 2$ for all x, y, and therefore $\alpha_1 \leq 4$, $\alpha_3 \leq 5$ and $\alpha_4 \leq 6$. Hence $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (4, 3, 5, 6)$. But $(*, *, 5, 6)$ was ruled out in the proof of Lemma 7.8.

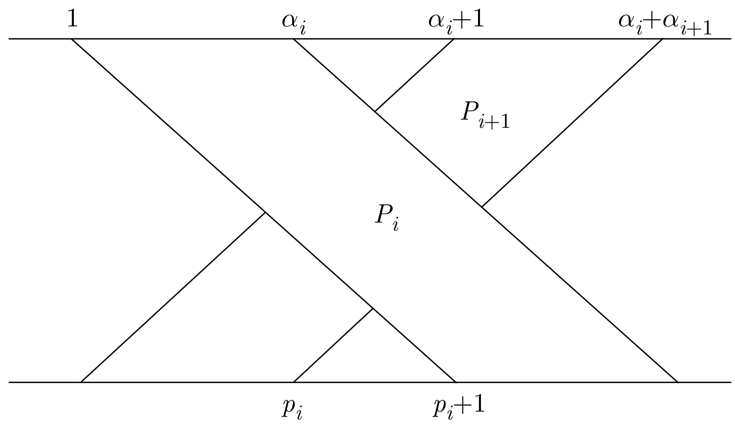


FIGURE 60

So suppose $\alpha_2 = 0$. The cases $(7,0,8,*)$, $(*,0,8,9)$, $(*,0,5,6)$, $(*,0,2,9)$, and $(4,0,*,9)$ were ruled out in the proof of Lemma 7.8. Since $\alpha_1 = 1,4$, or 7, $\alpha_3 = 2,5$ or 8, and $\alpha_4 = 3,6$, or 9, it is easy to check that this leaves only (4,0,8,6) and (7,0,5,9). But these were also ruled out in the proof of Lemma 7.8. □

CASE (4). $n_2 \geq 4$. Write $n_2 = n$.

Lemma 7.11. *$\alpha_i \leq 2n$, $i = 1,2,3,4$.*

Proof. This follows from Corollary 5.5. □

Corollary 7.12. *$\Delta \leq 7$.*

Proof. $\sum \alpha_i = \Delta n$, and hence $\Delta \leq 8$ by Lemma 7.11. Also, if $\Delta = 8$ then $\alpha_i = 2n$, $i = 1,2,3,4$, and therefore $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3. □

Our next goal is to prove Lemma 7.15. First we prove the following (essentially weaker) lemma, which will help us to eliminate certain small cases in Lemma 7.15.

Lemma 7.13. *If $\alpha_i + \alpha_j > 3n$ then $\pi_i = \pi_j^{\pm 1}$.*

Proof. None of the edges in P_i are parallel in Γ_2 by Lemma 2.1, and similarly for P_j. Hence if $\pi_i \neq \pi_j^{\pm 1}$, the edges in P_i and P_j define a subgraph Λ of Γ_2 with n vertices, $E = \alpha_i + \alpha_j$ pairwise non-parallel edges, and F faces, say. Since $\chi(\widehat{F}_2) = 0$ we have $F = E - n$, and since each face of Λ has at least three sides we have $2E \geq 3F$. Hence $E \leq 3n$. □

The following lemma will also be used to eliminate some of the small cases in Lemma 7.15.

Lemma 7.14. *(i) $\pi_i = \pi_{i+1}$ if and only if $\alpha_i + \alpha_{i+1} \equiv 0 \pmod n$*
(ii) $\pi_i = \pi_{i+2}$ if and only if $(\alpha_i + \alpha_{i+1}) + (\alpha_{i+1} + \alpha_{i+2}) \equiv 0 \pmod n$.

Proof. Recall that $\pi_i(x) \equiv x + p_i \pmod n$, for some p_i, $1 \leq i \leq 4$. From Figure 60 we see that $p_{i+1} \equiv p_i - (\alpha_i + \alpha_{i+1}) \pmod n$. The result follows. □

Lemma 7.15. *If $\alpha_i, \alpha_j \geq n+1$ then $\pi_i = \pi_j^{\pm 1}$.*

Proof. First we show that we may assume that some $\alpha_i \geq n+5$. For if not, then $\Delta n = \sum \alpha_i \leq 4(n+4)$, giving $(\Delta - 4)n \leq 16$. Therefore either $\Delta = 8$ and $n = 4$, or $\Delta = 7$ and $n = 4$ or 5, or $\Delta = 6$ and $n = 4, 5, 6, 7$ or 8. Now if $n = 4$ or 6, there is up to sign only one possible permutation with a single orbit, so the lemma is trivially true in this case.

First consider the case $\Delta = 7$, $n = 5$. Assuming that each $\alpha_i \leq n + 4 = 9$, the only possibility for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ (up to cyclic permutation) is (9,9,9,8). The result now follows from Lemma 7.13.

If $\Delta = 6$, $n = 8$, and each $\alpha_i \leq n+4 = 12$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (12, 12, 12, 12)$. Hence $\pi_1 = \pi_2 = \pi_3 = \pi_4$ by Lemma 7.14(i).

If $\Delta = 6$, $n = 7$, and each $\alpha_i \leq n + 4 = 11$, then the only possibilities for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ (up to cyclic permutation) are (11,11,11,9), (11,11,10,10) and (11,10,11,10). In the first case, $\pi_1 = \pi_2^{\pm 1} = \pi_3^{\pm 1}$ by Lemma 7.13. Hence either $\pi_1 = \pi_2$, or $\pi_2 = \pi_3$, or $\pi_1 = \pi_3$. But this contradicts Lemma 7.14. In the second case, $\pi_1 = \pi_2^{\pm 1}$ by Lemma 7.13, and by Lemma 7.14(i), $\pi_2 = \pi_3$ and $\pi_1 = \pi_4$ (and in fact $\pi_1 = \pi_2^{-1}$). Finally, in the case (11,10,11,10) $\pi_1 = \pi_2 = \pi_3 = \pi_4$ by Lemma 7.14(i).

Finally, if $\Delta = 6$, $n = 5$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ must be some permutation of (9,9,9,3), (9,9,8,4), (9,9,7,5), (9,9,6,6), (9,8,8,5), (9,8,7,6), (9,7,7,7), (8,8,8,6) or (8,8,7,7). In all cases except (9,9,6,6), (9,8,7,6), (8,8,8,6) and (8,8,7,7), the desired conclusion follows immediately from Lemma 7.13. In these remaining cases, Lemmas 7.13 and 7.14 give either the desired conclusion or a contradiction. For example, if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (9, 7, 8, 6)$ then by Lemma 7.14(i) we have $\pi_1 = \pi_4$, $\pi_2 = \pi_3$, and by Lemma 7.13, $\pi_1 = \pi_2^{\pm 1}$. On the other hand, for the case (9,8,6,7), Lemmas 7.13 and 7.14(i) imply that $\pi_1 = \pi_2^{-1}$, $\pi_1 = \pi_4^{-1}$, hence $\pi_2 = \pi_4$. But this is impossible by Lemma 7.14(ii). The other cases follow similarly.

We suppose, then, that we have $n + 5$ parallel edges of Γ_1, say $A_1, A_2, \ldots, A_n$, B_1, B_2, B_3, B_4, B_5, in anticlockwise order around vertex $+$, with associated permutation π. Let N be the annulus obtained by cutting $\widehat{F}_2$ along $A_1, A_2, \ldots, A_n$ in Γ_2. We will use $\boldsymbol{\ell}$ to denote the vertex $\pi^{\ell}(1)$ of Γ_2. As mentioned above, we may assume that $n \geq 5$.

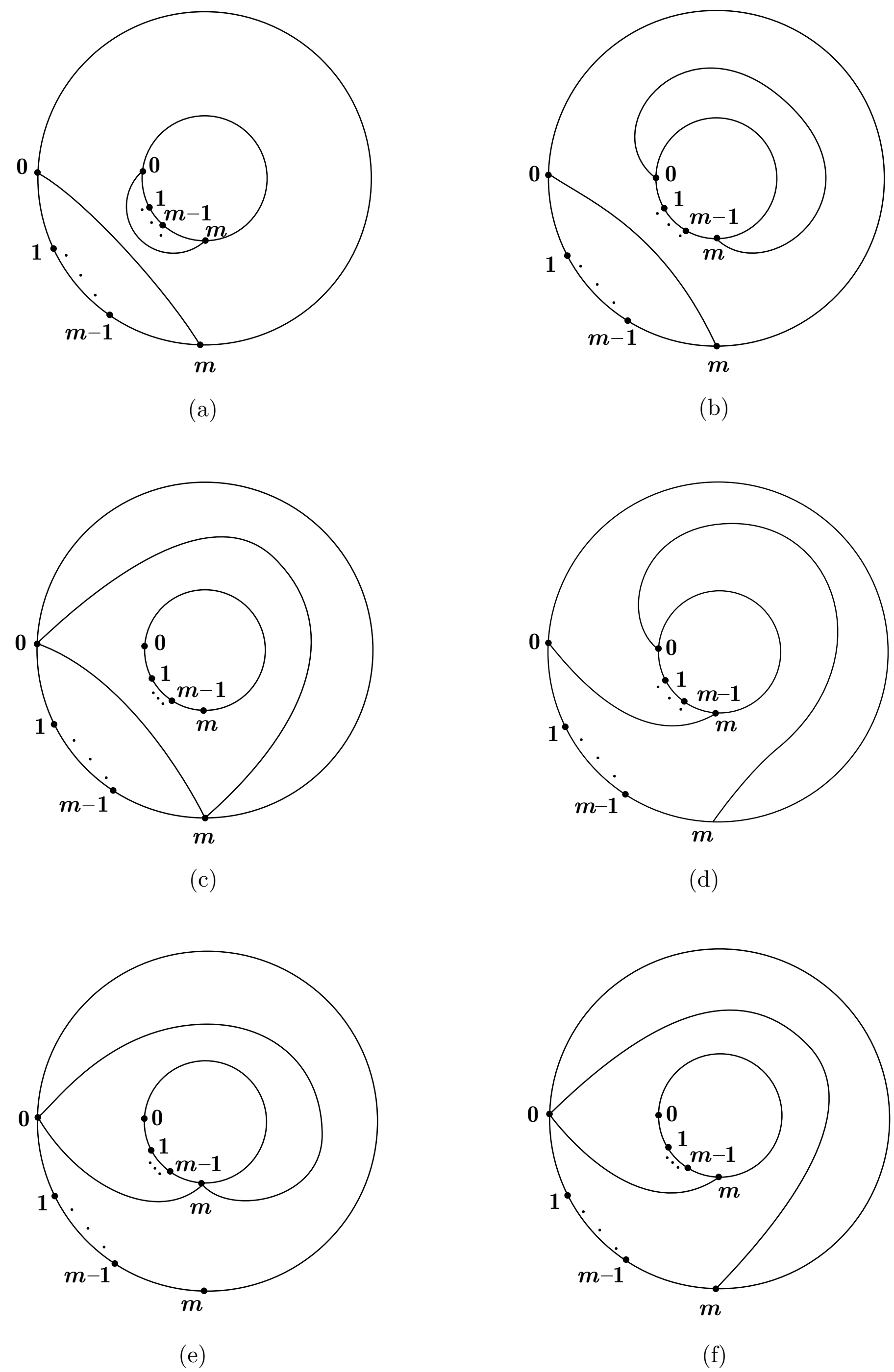
Suppose $\alpha_i \geq n + 1$. We shall show that $\pi_i = \pi^{\pm 1}$. So assume not. Then (since π has only one orbit by Lemma 4.2) $\pi_i = \pi^m$ where $m \neq \pm 1$. Then P_i contains edges X and Y, each of which has (without loss of generality) label $\mathbf{0}$ at vertex $+$ and label $\boldsymbol{m}$ at vertex $-$.

The essentially different possible arrangements of X and Y in N are shown in Figure 61(a)–(f). The cases where X and Y are both of type I (as defined in the proof of Lemma 5.4) are shown in (a), (b), and (c); when both are of type II in (d) and (e), and when one is of type I and the other of type II, in (f).

First we dispose of case (a) by noting that it clearly precludes the existence of the edge in P_i with label $\mathbf{1}$ at $+$ and $\boldsymbol{m+1}$ at $-$.

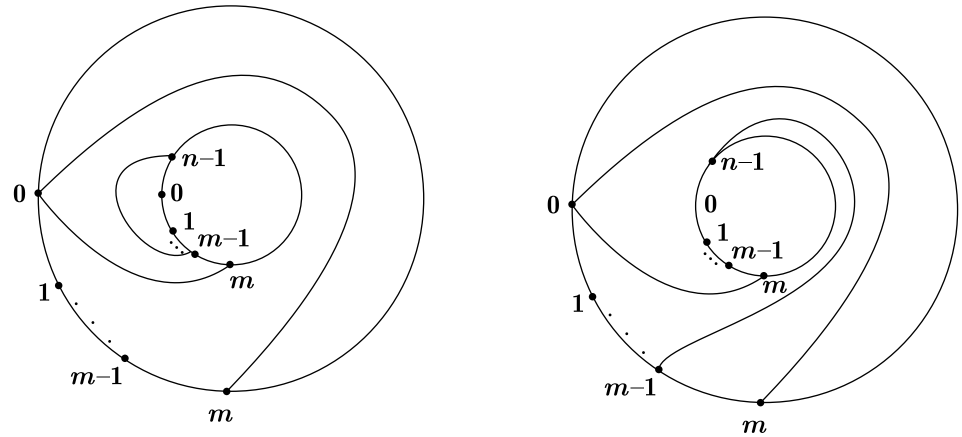
Now consider the edges $B_1, \ldots, B_5$. The edge B_j joins vertices $\boldsymbol{k_j}$, $\boldsymbol{k_j+1}$ of Γ_2, for some k_j, and of course B_j is not parallel to A_j in Γ_2 by Lemma 2.1, $1 \leq j \leq 5$.

Note that if B_1 (say) is of type I, then firstly, there is at most one other B_j of type I (with its endpoints on the other component of ∂N), and secondly, any B_j of type II must share an endpoint with B_1. It follows easily that B_j is of type II, $1 \leq j \leq 5$.



0
0
1
m−1
m
1
m−1
m
(a)
0
0
1
m−1
m
1
m−1
m
(b)
0
0
1
m−1
m
1
m−1
m
(c)
0
0
1
m−1
m
1
m−1
m
(d)
0
0
1
m−1
m
1
m−1
m
(e)
0
0
1
m−1
m
1
m−1
m
(f)

Figure 61



(i) (ii)

FIGURE 62

Let the boundary components of N be $\partial_0 N$ and $\partial_1 N$, illustrated in the figures as the outer and inner boundary components, respectively. Let $(\boldsymbol{k}, \boldsymbol{\ell}\,)$ denote the pair of vertices $\boldsymbol{k}$ on $\partial_0 N$ and $\boldsymbol{\ell}$ on $\partial_1 N$. Thus the pair of endpoints of B_j is either $(\boldsymbol{k}_j, \boldsymbol{k_j}+\boldsymbol{1})$ or $(\boldsymbol{k_j}+\boldsymbol{1}, \boldsymbol{k_j})$. It is straightforward to check that in each of the cases (b), (c), (d) or (e) there can be at most four such edges disjoint from X and Y. For instance, in case (b) the possible endpoint pairs are $(\boldsymbol{n-1}, \boldsymbol{0})$, $(\boldsymbol{0}, \boldsymbol{1})$, $(\boldsymbol{m}, \boldsymbol{m-1})$, and $(\boldsymbol{m+1}, \boldsymbol{m})$. The other three cases are similar.

Finally, consider case (f). Let Z be the edge in P_i with label $\boldsymbol{n-1}$ at vertex $+$ and lable $\boldsymbol{m-1}$ at vertex $-$. As it lies in N, Z is either of type I or type II; the two possibilities are illustrated in Figures 62(i) and (ii) respectively. It is now easy to check that in each case there are at most four possible disjoint edges, disjoint from X, Y and Z, with endpoints of the form $(\boldsymbol{k}, \boldsymbol{k+1})$ or $(\boldsymbol{k+1}, \boldsymbol{k})$. □

In the sequel, π will denote the permutation π_i associated with some P_i with $\alpha_i \geq n+1$.

Lemma 7.16. *$\nu(x, \pi(x)) = 2$ for all vertices x of Γ_2.*

Proof. First note that we may suppose that $\alpha_i \leq n$ for at most one value of i. For suppose $\alpha_1, \alpha_2 \leq n$ (say). Then $\alpha_3 + \alpha_4 \geq (\Delta - 2)n$. Since $\alpha_3 + \alpha_4 \leq 4n$ by Lemma 7.11, this is a contradiction unless $\alpha_1 = \alpha_2 = n$, $\alpha_3 = \alpha_4 = 2n$ (and $\Delta = 6$). But this implies that $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3.

So we may suppose that $\alpha_1, \alpha_2, \alpha_3, \geq n+1$. Hence $\pi_i = \pi^{\pm 1}$, $i = 1, 2, 3$, by Lemma 7.15.

If $\alpha_4 \geq n+1$, then $\pi_4 = \pi^{\pm 1}$ also. Since not all the π_i's can be equal, by Lemma 7.3, we have (without loss of generality) that either $\pi_1 = \pi_2 = \pi_3^{-1} = \pi_4^{-1}$, or $\pi_1 = \pi_2 = \pi_3 = \pi_4^{-1}$. In the first case, we may again suppose without loss of generality that $\alpha_1 + \alpha_2 \geq 3n$, and thus every vertex x of Γ_2 appears twice in either P_1^+ or P_2^+. (Here P_i^+ denotes the labels at the endpoints at vertex $+$ of the edges in P_i.) Therefore $\nu(x, \pi(x)) \geq 2$ by Lemma 2.1. Hence $\nu(x, \pi(x)) = 2$ for all x, by Lemma 5.4(ii). In the second case, since $\alpha_1 + \alpha_2 + \alpha_3 \geq 4n$ by Lemma 7.11, every vertex x of Γ_2 appears twice in either P_1^+, P_2^+ or P_3^+. Therefore $\nu(x,\pi(x)) = 2$ for all x, as before.

Now suppose $\alpha_4 \leq n$. Since $\pi_i = \pi^{\pm 1}$, $i = 1, 2, 3$, we may suppose without loss of generality that $\pi_1 = \pi_2 = \pi$. Since $\alpha_3 \leq 2n$ by Lemma 7.11, $\alpha_1 + \alpha_2 \geq 3n$. Therefore $\nu(x,\pi(x)) = 2$ for all vertices x of Γ_2, as above. □

Lemma 7.17. *If $\Delta = 7$ then no three permutations π_i are equal.*

Proof. Suppose that $\pi_1 = \pi_2 = \pi_3 = \pi$. Since $\alpha_4 \leq 2n$, we have $\alpha_1 + \alpha_2 + \alpha_3 \geq 5n$. Therefore, for all vertices x of Γ_2, there are at least five edges in Γ_1 with label x at $+$ and label $\pi(x)$ at $-$. Since $\nu(x,\pi(x)) = 2$ by Lemma 7.16, some three of these edges must belong to the same parallelism class in Γ_2. Since these edges all have label $+$ at x in Γ_2, this parallelism class would then contain at least five edges, contradicting (by Lemma 2.1) the fact that Γ_1 has only four parallelism classes of edges. □

Lemma 7.18. $\Delta = 7$ *is impossible.*

Proof. First note that we may assume that $\alpha_i \geq n + 1$, $i = 1, 2, 3, 4$. For if $\alpha_1 = n$ (say), then (by Lemma 7.11) we must have $\alpha_2 = \alpha_3 = \alpha_4 = 2n$. This implies that $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3.

By Lemma 7.15 we therefore have $\pi_i = \pi^{\pm 1}$, $i = 1, 2, 3, 4$. By Lemma 7.17, two of these are π and two are π^{-1}. Let x be a vertex of Γ_2. Then, without loss of generality, at vertex $+$ of Γ_1 the label x appears exactly twice in each of the parallelism classes of edges with permutation π^{-1}, exactly twice in one of the parallelism classes with permutation π, and exactly once in the remaining parallelism class with permutation π.

Consider the corresponding edges in Γ_2, i.e., those with label $+$ at vertex x. Note that in Γ_2, any edge incident to x joins x to either $\pi(x)$ or $\pi^{-1}(x)$, and, by Lemma 7.16, $\nu(x,\pi(x)) = \nu(x,\pi^{-1}(x)) = 2$. Hence, by Lemma 2.1, of the edges with label $+$ at x, exactly two belong to each of the parallelism classes joining x to $\pi^{-1}(x)$, exactly two belong to one of the parallelism classes joining x to $\pi(x)$, and exactly one to the other such parallelism class. Let these last three edges be A, B and C respectively. Then around vertex x these edges appear as in Figure 63(i), (ii) or (iii).

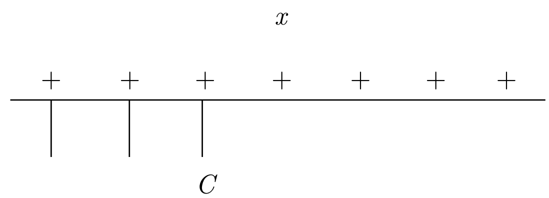
A similar argument applied to the edges of Γ_2 with label $-$ at vertex $\pi(x)$ shows that the edges A, B and C appear at vertex $\pi(x)$ as in Figure 64(i), (ii) or (iii). (These possibilities are independent of the possible arrangements at vertex x.)

Now in Γ_1, C is parallel to either A or B. On the other hand, from Figures 63 and 64 we see that if $A(+)$ denotes the endpoint of A with label $+$, etc., then (with suitable choice of orientation) we have $\delta_2(A(+), C(+)) = 4, 8$ or 12, $\delta_2(A(-), C(-)) = 10, 6$ or 2, and $\delta_2(B(+), C(+)) = 2, 6$ or 10, $\delta_2(B(-), C(-)) = 12, 8$ or 4. This contradicts Lemma 2.5(i). □

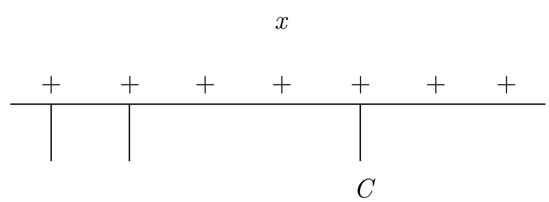
Lemma 7.19. $\Delta = 6$ *is impossible.*

Proof. First note that at most one α_i is $\leq n$. For if $\alpha_1, \alpha_2 \leq n$ (say), then $\alpha_3 + \alpha_4 \geq 4n$, and therefore, by Lemma 7.11, $\alpha_1 = \alpha_2 = n$, and $\alpha_3 = \alpha_4 = 2n$. But this implies that $\pi_1 = \pi_2 = \pi_3 = \pi_4$, contradicting Lemma 7.3.

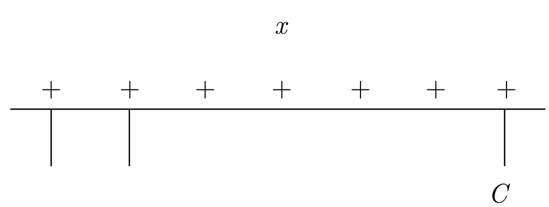
So suppose $\alpha_1, \alpha_2, \alpha_3 \geq n + 1$. Then $\pi_i = \pi^{\pm 1}$, $i = 1, 2, 3$. Note that $\alpha_4 > 0$, for if not then we would have $\alpha_1 = \alpha_2 = \alpha_3 = 2n$, contradicting (the proof of) Lemma 7.3. Also (by re-choosing α_4 if necessary) we may assume that $\alpha_4 < 2n$.



(i)



(ii)



(iii)

FIGURE 63

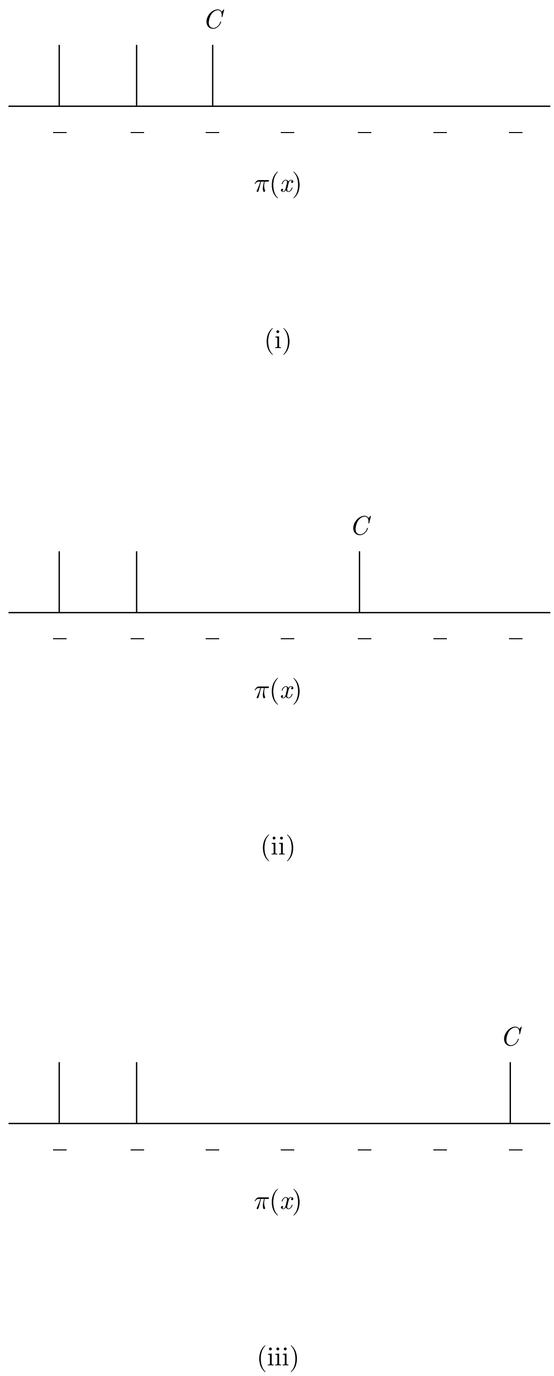


Figure 64

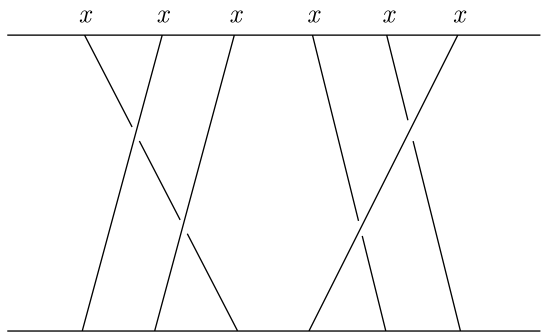


FIGURE 65

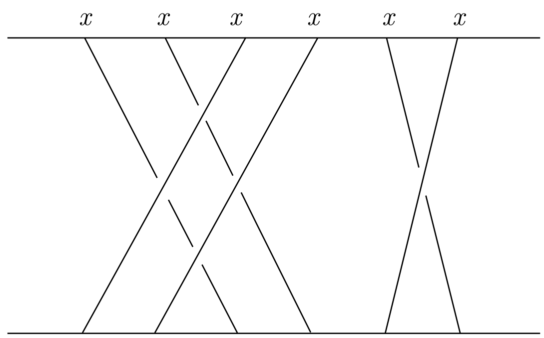


FIGURE 66

Let x be a label that appears exactly once in P_4^+. Then x appears exactly once in one of P_1^+, P_2^+, P_3^+, and exactly twice in the two others. The three possible arrangements are shown in Figures 65, 66 and 67. In particular, there are at least three edges in $P_1 \cup P_2 \cup P_3$ with label x at $+$ and (say) $\pi(x)$ at $-$. By Lemma 7.16, two of these, say A and B, are parallel in Γ_2. Since the number of edges in any parallelism class in Γ_2 is at most four, A and B are adjacent around x among the edges of Γ_2 with label $+$ at x. Since $d = 1$, A and B are adjacent at $+$ among those edges in Γ_1 with label x at $+$. But examination of Figures 65, 66 and 67 shows that any such adjacent pair of edges violates either Lemma 2.1 or Lemma 2.5(i). $\square$

We have thus shown that in Case (B), the only possibilities for F_1, F_2 are given by the identification patterns $P(7)$ and $P(6)_2$.

8. CASE (C)

In this section we treat the case where $\Delta = 6$, $n_1 \geq 3$, $n_2 = 2$, the two boundary components of F_2 are of opposite sign, and F_2 is bad.

It follows from Lemma 3.2 that the arcs of $F_1 \cap F_2$ in F_2 come in parallelism classes of size n_1. Thus $\Gamma_2 \cong G(n_1, n_1, n_1, n_1, n_1)$. Call the boundary components of F_2 (vertices of Γ_2) $+$ and $-$. The n_1 loops at vertex $+$ in Γ_2 define a permutation π of $\{1, 2, \ldots, n_1\}$ with orbits of size 2 (thus n_1 is even). Since $n_1 \geq 3$, there are at least two such orbits. This gives edges in Γ_1 as shown in Figure 68. Write $n_1 = 2n$.

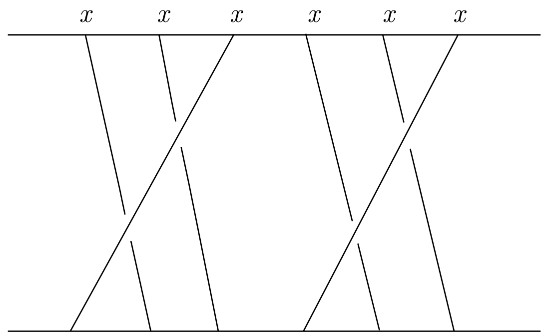


FIGURE 67

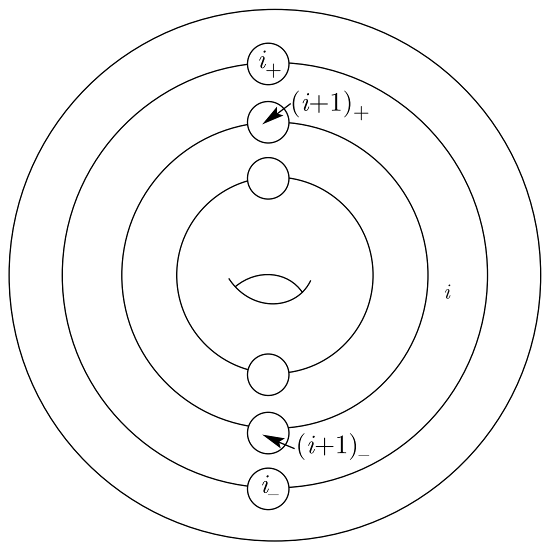


FIGURE 68

The four parallelism classes of edges in Γ_2 that run from $+$ to $-$ all define the same permutation ρ; that is, if an edge has label i at vertex $+$, then it has label $\rho(i)$ at vertex $-$. Note that the elements in any orbit of ρ all have the same sign, and that the corresponding cycle must be essential in $\widehat{F}_1$ by Lemma 2.3.

Let the orbits of π be $\{i_+, i_-\}$, $i = 1, 2, \dots, n$. The corresponding edges of $\overline{\Gamma}_1$ decompose $\widehat{F}_1$ into annuli $N_1, N_2, \dots, N_n$, numbered so that ∂N_i contains the four vertices $i_\pm$, $(i+1)_\pm$ of Γ_1 (see Figure 68). The only vertices of the same sign accessible from i_+ are $(i \pm 1)_+$ and i_+. Hence either

(1) ρ is the identity, or
(2) $\rho(i_+) = (i+1)_+$, say, and $\rho(i_-) = (i+\varepsilon)_-$, where $\varepsilon = \pm 1$, so ρ has exactly two orbits $\{i_+ : i = 1, 2, \dots, n\}$ and $\{i_- : i = 1, 2, \dots, n\}$.

In case (1), consider one of the corresponding loops in Γ_1, at vertex i_+, say. This must lie in either N_i or N_{i-1}, and there cannot be such loops in both N_i and N_{i-1}, as this would prohibit the existence of a loop at i_-. Hence if we let $A_1, A_2, \dots, A_6$

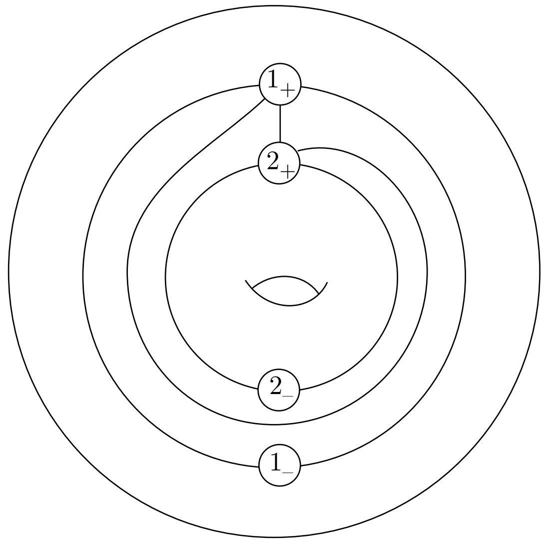


FIGURE 69

be the edges of Γ_1 with label $+$, say, at the vertex i_+, then they may be numbered consecutively around i_+ in such a way that A_1 and A_6 join i_+ to $i_- = \pi(i_+)$, and A_2, A_3, A_4, and A_5 join i_+ to $i_+ = \rho(i_+)$. Since $d = d_{12} = 1$, these edges appear in the same order around vertex $+$ in Γ_2. Thus, there, A_1 and A_6 are loops at $+$ while A_2, A_3, A_4, and A_5 join vertices $+$ and $-$. But this contradicts the form of Γ_2.

In case (2), we distinguish the two subcases (a) $n_1 \geq 6$, and (b) $n_1 = 4$.

First consider (a). Here, any edge of $\overline{\Gamma}_1$ joining vertices i_+ and $\rho(i_+) = (i+1)_+$ must lie in the annulus N_i. Hence there is at most one such edge, as otherwise there could be no edge joining i_- and $(i+1)_-$. The four edges of Γ_2 with label i_+ at $+$ and $\rho(i_+)$ at $-$ are therefore all parallel in Γ_1. But this implies the existence of an edge in Γ_1 with label $-$ at i_+ and label $+$ at $\rho(i_+)$. Considering this edge in Γ_2 shows that ρ^2 is the identity, contradicting our assumption that $n_1 \geq 6$.

Finally, consider subcase (b). Let X_1, X_2 be two edges of $\overline{\Gamma}_1$ joining vertices 1_+ and 2_+ that come from some parallelism class in Γ_2 with corresponding permutation ρ. If these both lie in N_1, say (see Figure 69), then the corresponding edges joining 1_- and 2_- must lie in N_2, and hence X_1, X_2 are the only edges of $\overline{\Gamma}_1$ joining 1_+ and 2_+. Thus the arrangement of the edges $A_1, A_2, \ldots, A_6$ in Γ_1 with label $+$ at vertex 1_+ around that vertex is as described in case (1) above, giving the same contradiction.

If X_1, X_2 lie in N_1, N_2 respectively, then $\overline{\Gamma}_1$ is as shown in Figure 70. Choose the numbering of the labels around vertex $+$ of Γ_2 so that each parallelism class of edges has labels 1,2,3,4 there, in anticlockwise order. Then we must have $\rho(i) \equiv i + 2 \pmod 4$, and so the labels in any parallelism class at vertex $-$ are (in clockwise order) 3,4,1,2. Now let A be one of the loops at vertex $+$ in Γ_2 with endpoint labels 1 and 4, and let X, Y be the two loops at vertex $-$ with endpoint labels 1 and 4. See Figure 71. Then, since there are exactly two parallelism classes of edges in Γ_1 joining vertices 1 and 4, A must be parallel in Γ_1 to either X or

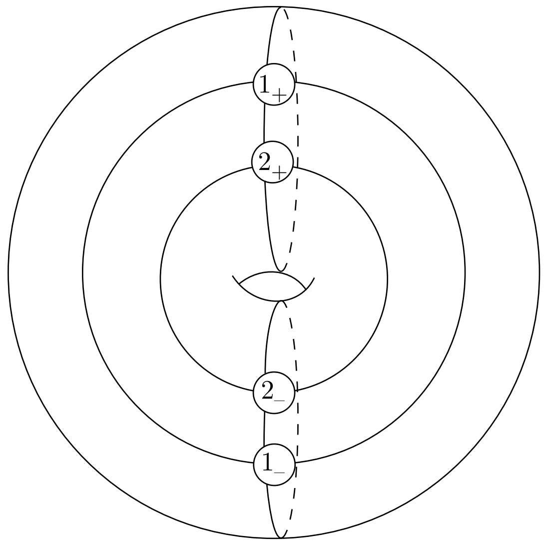


FIGURE 70

Y. But if $A(i)$ denotes the endpoint of A in Γ_2 with label i, and similarly for X and Y, then we have (for suitable choice of orientation) $\tau_2(A(1), A(4)) = 15$, $\tau_2(X(1), X(4)) = \tau_2(Y(1), Y(4)) = 11$. This contradicts Lemma 2.5(iii).

We have thus shown that Case (C) cannot occur.

9. CASE (D)

In this section we treat the case where $\Delta = 6$, $n_1 \geq 3$, all the boundary components of F_2 have the same sign, and F_2 is bad.

It follows from Lemma 3.2 that the edges of Γ_2 come in parallelism classes of size n_1. Each such parallelism class defines a permutation ρ with orbits of size 2. Thus n_1 is even and the number of orbits of ρ is $n_1/2 \geq 2$. The parallelism class gives rise to edges in the reduced graph $\overline{\Gamma}_1$ as shown in Figure 68. It is then easy to see that $\overline{\Gamma}_1$ is a subgraph of either the graph illustrated in Figure 72, or, when $n_1 = 4$, the graph illustrated in Figure 73. In particular, there are at most two permutations ρ.

We distinguish the three cases $n_2 = 1$, $n_2 = 2$, and $n_2 \geq 3$.

CASE (1). $n_2 = 1$. Here $\Gamma_2 \cong H(n_1, n_1, n_1)$, and there is only one permutation ρ. Thus $\overline{\Gamma}_1$ is as shown in Figure 68. Since $d = 1$, this determines the identification between the edges of Γ_1 and Γ_2 (modulo permutation of the orbits of ρ on $\widehat{F}_1$), namely that given by the identification pattern $P(6)_{2n}$ (where $n_1 = 2n$), $n \geq 2$, shown in Figure 74. Although this case is combinatorially possible, we shall see in Section 11 that it is actually topologically degenerate.

CASE (2). $n_2 = 2$. Here $\Gamma_2 \cong G(n_1, n_1, n_1, n_1, n_1)$. Let the vertices of Γ_2 be x and y. Without loss of generality, suppose that the labels at the ends of any parallelism class of edges at the vertex x are, in order, $1, 2, \ldots, n_1$. Let the labels at the vertex y be $m, m+1, \ldots, m-1$. Then the permutations defined by the loops

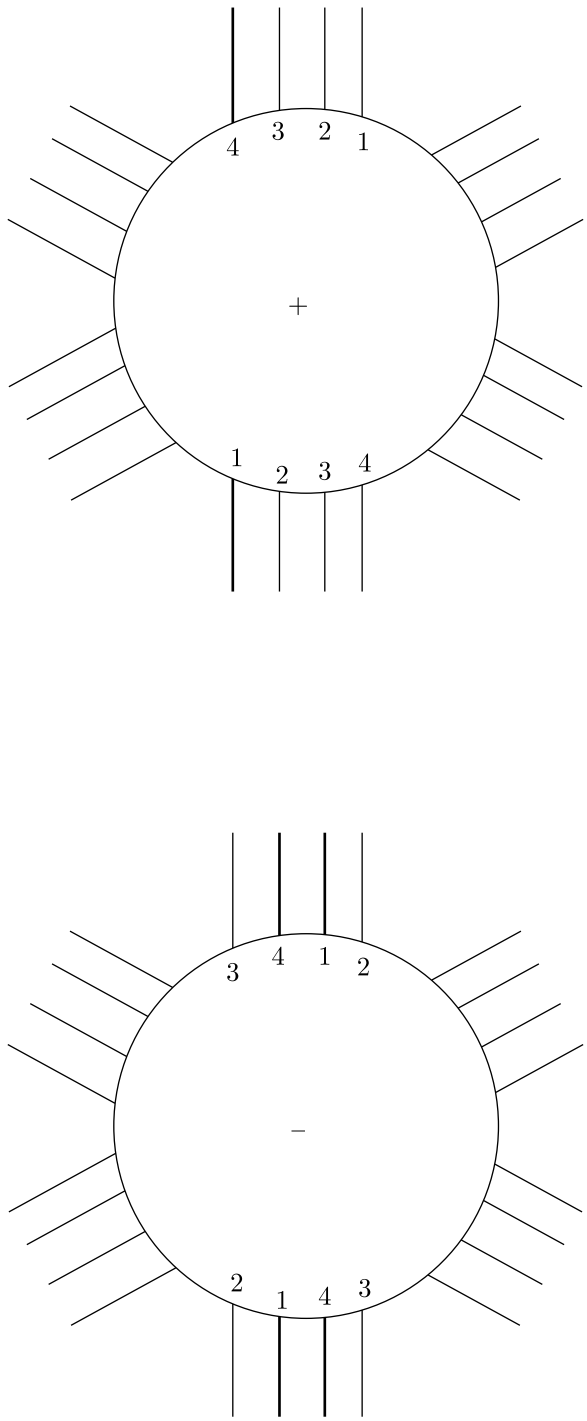


Figure 71

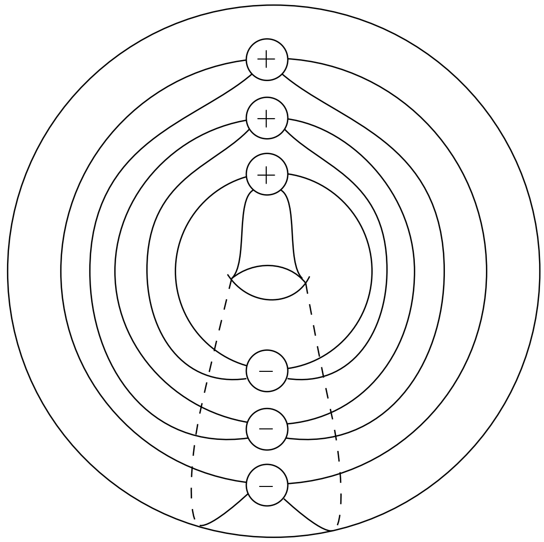


FIGURE 72

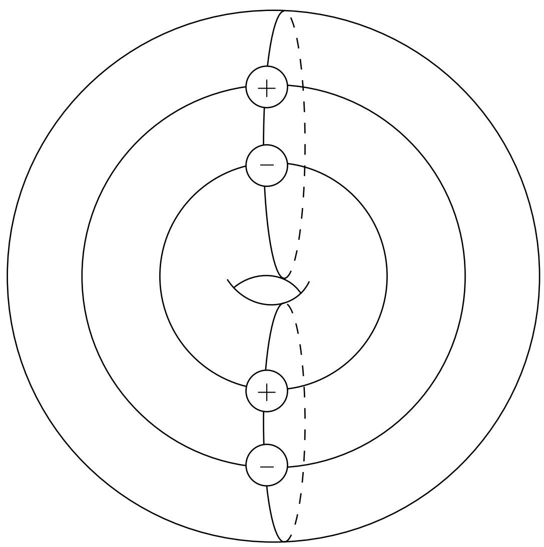
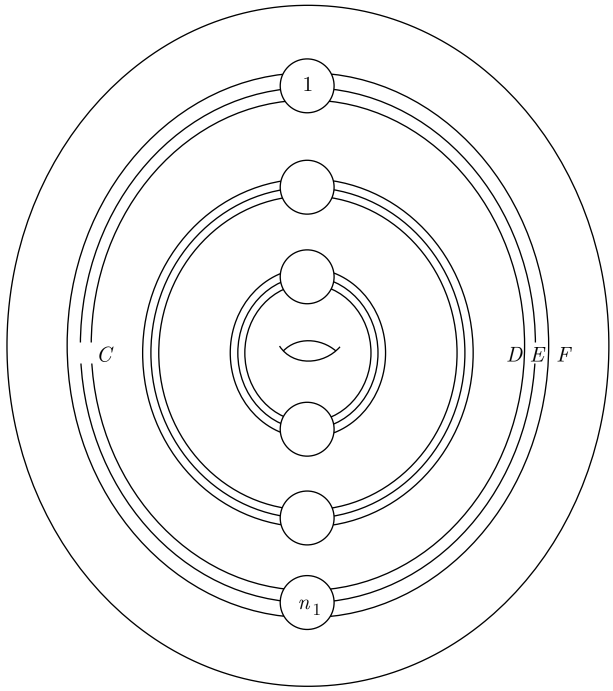


FIGURE 73

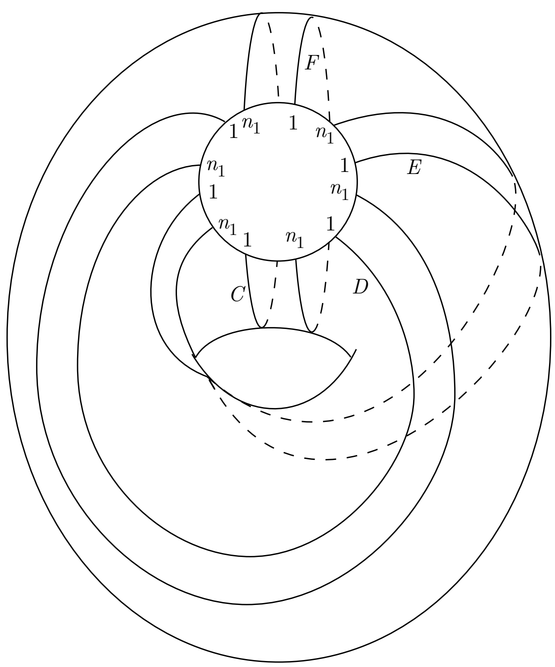
at x, the loops at y, and the edges joining x and y, respectively, are

$$\rho_1 : i \mapsto 1 - i,$$
$$\rho_2 : i \mapsto 2m - 1 - i,$$
$$\rho_3 : i \mapsto m - i\ .$$

By the remarks at the beginning of the section, at least two of these permutations must coincide.



(i)



(ii)

FIGURE 74

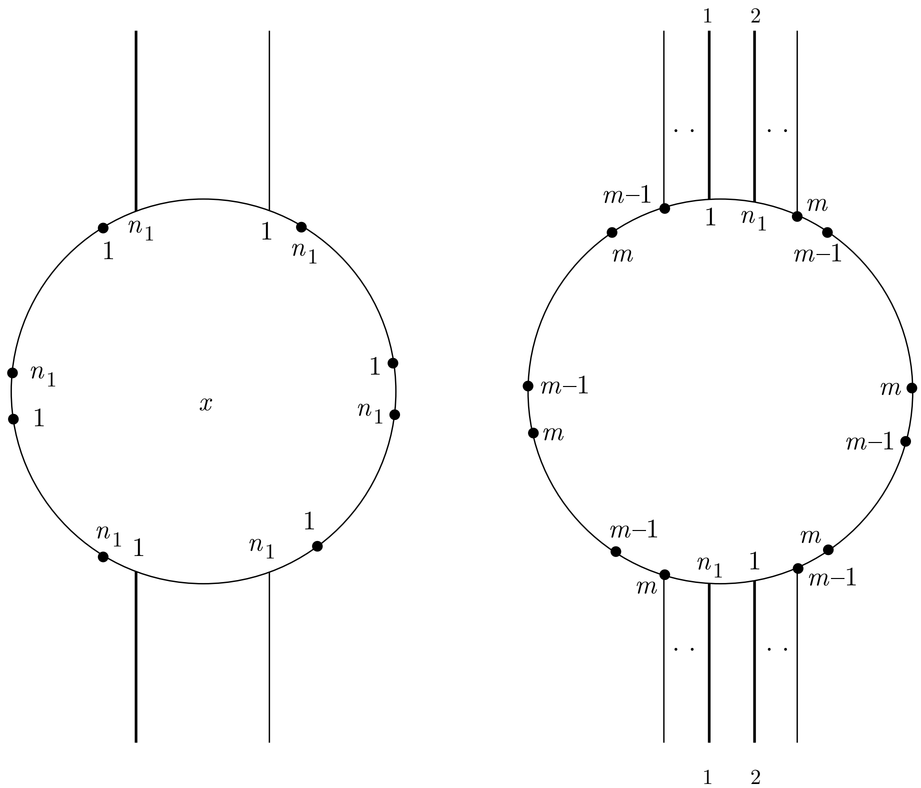


FIGURE 75

First suppose that all three permutations are equal, that is, $m = 1$. A loop in Γ_2 gives rise to an edge in Γ_1 joining vertices i and $1-i$, say, with equal labels at its two endpoints. But this implies (see Figure 68) that every edge in Γ_1 has equal labels at its endpoints, *i.e.*, that Γ_2 consists entirely of loops. This is a contradiction.

If exactly two permutations coincide, then it is easy to see that they must be ρ_1 and ρ_2, and $m = (n_1/2) + 1$. Let X be one of the loops in Γ_2 based at x with endpoint labels 1 and n_1, and let Y_1, Y_2 be the loops based at y with endpoint labels 1 and n_1 (see Figure 75). Since there are only two parallelism classes of edges in Γ_1 joining vertices 1 and n_1, X must be parallel in Γ_1 to either Y_1 or Y_2. But this is impossible by Lemma 2.5(iii), since, if $X(i)$ denotes the endpoint of X in Γ_2 with label i, and similarly for Y_1 and Y_2, then $\tau_2(X(1), X(n_1)) = 4n_1 - 1$ (say), while $\tau_2(Y_j(1), Y_j(n_1)) = 3n_1 - 1$, $j = 1, 2$.

CASE (3). $n_2 \geq 3$. First suppose that there is only one permutation ρ. Then the edges in Γ_1 joining any vertex a to $\rho(a)$ fall into two parallelism classes, each of size $3n_2$. If $n_2 \geq 4$, this contradicts Corollary 5.5. In any case, each of the parallelism classes defines a permutation π of $\{1, 2, \ldots, n_2\}$ which by Lemma 4.2 has only one orbit. Let $A_1, A_2, A_3, B_1, B_2, B_3$ be the edges with label x at some vertex a, and label $\pi(x)$ at $\rho(a)$, for some x, numbered cyclically around a in such a way that the A_i's belong to one parallelism class and the B_i's to the other. Since $d = 1$, these edges appear in the same order around the vertex x in Γ_2. Since $\nu(x, \pi(x)) \leq 5$ by

Lemma 5.4(i), some consecutive pair must be parallel in Γ_2. But this is impossible by Lemmas 2.1 and 2.5(i).

Finally, suppose that there are two permutations ρ, say ρ_1 and ρ_2. First note that since the valency of each vertex a in $\overline{\Gamma}_1$ is 4, some parallelism class of edges in Γ_1 joining a to $\rho_1(a)$, say, has at least n_2+1 members. By Lemma 4.2, the corresponding permutation π has only one orbit. Let the first n_2 of these edges be $A_1, A_2, \dots, A_{n_2}$, where (without loss of generality) A_i has label i at a and label $\pi(i)$ at $\rho_1(a)$.

For any vertex of Γ_2, the labels at the end of each parallelism class of edges incident at that vertex are, in order, $m, m+1, \dots, m-1$ for some m. We may suppose that at vertex 1, say, $m=1$. If $m=1$ for all vertices of Γ_2, then we get only one permutation ρ, contrary to hypothesis. Since the edges $A_1, A_2, \dots, A_{n_2}$, as they lie in Γ_2, have label a at vertex i and label $\rho_1(a)$ at $\pi(i)$ for all i, we see that, at the vertex $\pi^k(1)$, $m=1$ if k is even and $m=m_0$ if k is odd, where $m_0 \neq 1$ is independent of k. In particular it follows that n_2 is even, and hence ≥ 4.

Since this argument applies to any parallelism class of edges in Γ_1 with at least n_2+1 members, it also follows that there can be no such parallelism class joining vertices a and $\rho_2(a)$. Hence if the two parallelism classes joining a to $\rho_i(a)$ each have α_i members, $i=1,2$, then $\alpha_2 \leq n_2$. Also, $2\alpha_1 + 2\alpha_2 = 6n_2$. Hence, $\alpha_2 < n_2$ implies $\alpha_1 > 2n_2$, contradicting Corollary 5.5. We must therefore have $\alpha_1 = 2n_2$, $\alpha_2 = n_2$. It follows that any edge in Γ_1 joining a and $\rho_1(a)$ has label i at a and label $\pi(i)$ at $\rho_1(a)$, say. But from the form of Γ_2 the edges in Γ_1 joining a and $\rho_1(a)$ come in pairs, one with label i at a and label $\pi(i)$ at $\rho_1(a)$ and the other with label $\pi(i)$ at a and label i at $\rho_1(a)$. This implies that π^2 is the identity, and hence $n_2 = 2$, contrary to hypothesis.

We have thus shown that in Case (D), the only possibilities for F_1, F_2 are given by the identification patterns of the form $P(6)_{2n}$, $n \geq 2$.

10. Case (E)

In this section we treat the case where $\Delta = 6$ and F_1 and F_2 are bad.

By Lemma 3.2, the edges of Γ_α come in parallel families of size n_β. Also, since (again by Lemma 3.2) Γ_α has a face that is a disk with three sides, there is a family of n_β parallel edges in Γ_α joining boundary components of the same sign. The orbits of the corresponding permutation each contain two vertices of Γ_β, of opposite sign. There are thus three cases:

(1) $n_1 = n_2 = 2$;
(2) $n_1 = 2$, $n_2 \geq 4$;
(3) $n_1, n_2 \geq 4$.

(1) is treated under Case (A), and (2) under Case (C), so it remains to consider (3).

By Lemma 2.3, a family of n_β parallel edges in Γ_α joining vertices of the same sign gives a set of edges in Γ_β as shown in Figure 68. Now consider a family of n_β parallel edges in Γ_α joining vertices of opposite sign. As in Section 8 (Case (C)), the corresponding permutation either is the identity or has exactly two orbits. Again arguing as in Section 8, we see that the number of edges of $\overline{\Gamma}_\beta$ joining vertices of the same sign is n_β. Since the edges of Γ_β come in parallel families of size n_α, this shows that the number of edges in Γ_β joining vertices of the same sign is $n_1 n_2$.

But the total number of edges is $\dfrac{6n_1n_2}{2} = 3n_1n_2$. Therefore, by the parity rule, either Γ_1 or Γ_2 has at least $\frac{3n_1n_2}{2}$ edges joining vertices of the same sign. This contradiction shows that case (3) cannot occur.

Hence Case (E) is impossible.

11. Final topological arguments

We have shown that if $\Delta \geq 6$ and (M, T) is not cabled, then, up to homeomorphism, the only possibilities for $(F_1, F_2; F_1 \cap F_2)$ are those described by the identification patterns $P(6)$, $P(6)_{2n}$, $P(7)$, $P(8)_1$, and $P(8)_2$, illustrated in Figures 28, 74, 35, 31, and 32. (Note that since in all cases the complementary regions on at least one of the surfaces are all disks, we may assume by a standard innermost circle argument that there are no circles of intersection.) In each of these cases, let $X = F_1 \cup F_2 \cup T$, and let $N(X)$ be an abstract regular neighborhood of $X \operatorname{rel} T$.

First we show that the family $P(6)_{2n}$ is topologically degenerate.

Lemma 11.1. *For $P(6)_{2n}$, $n \geq 1$, F_1 is compressible in $N(X)$.*

Proof. Let D_0 be a disk on T of the form shown in Figure 76, containing $\partial F_1 \cap \partial F_2$. We use D_0 as "base-point" for computations in $\pi_1(X)$. We choose D_0 so that the arc u on boundary component 1 of F_1 which does not lie in D_0 appears on F_1 as shown in Figure 77. Then the arc v on boundary component $2n$ shown in Figure 76 appears on F_1 as in Figure 77 (see Figure 74(ii)).

Let a, b, c, d, e, f be the elements of $\pi_1(X)$ represented by the arcs A, B, C, D, E, F, with the orientations indicated. Then, from the two 3-sided faces on F_2 (see Figure 74(ii)) we get the relations

$$ace = 1, \quad bdf = 1 \ ,$$

and from the two bigons on F_1 bounded by the arcs A, B and E, F respectively, we get the relations

$$a = b, \quad e = f \ .$$

Therefore $c = d$. But the curve γ shown in Figure 77 is essential on F_1 and represents cd^{-1} in $\pi_1(X)$. Hence the map $\pi_1(F_1) \to \pi_1(X)$ is not injective. □

Let $\partial_0 N(X)$ denote $\partial N(X) - T$.

Lemma 11.2. *For $P(6)$, $P(7)$, $P(8)_1$, and $P(8)_2$, every component of $\partial_0 N(X)$ is a 2-sphere.*

Note that $\chi(\partial_0 N(X)) = \chi(\partial N(X)) = 2\chi(N(X)) = 2\chi(X)$. We prove Lemma 11.2 by computing that in all four cases the number of components of $\partial_0 N(X)$ is $\chi(X)$; hence each must be a 2-sphere. We compute $\chi(X)$ easily as follows.

Lemma 11.3. $\chi(X) = \dfrac{\Delta n_1 n_2}{2} - n_1 - n_2$.

Proof. First note that

$$\begin{aligned}\chi(F_1 \cup F_2) &= \chi(F_1) + \chi(F_2) - \chi(F_1 \cap F_2) \\ &= -n_1 - n_2 - \frac{\Delta n_1 n_2}{2} \ .\end{aligned}$$

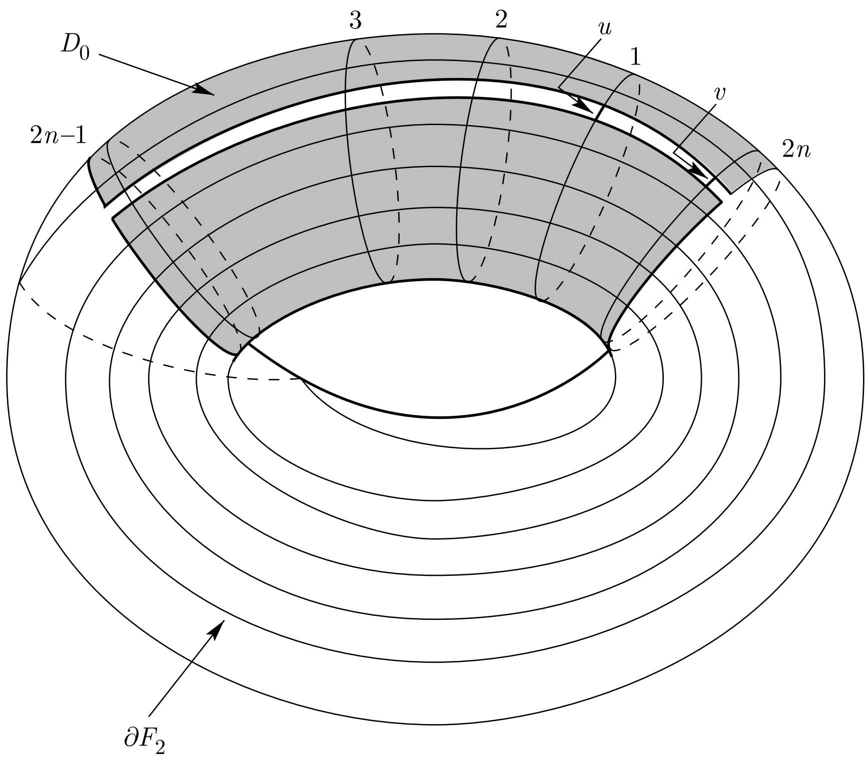


FIGURE 76

Also, $T \cap (F_1 \cup F_2) = \partial F_1 \cup \partial F_2$, and $\chi(\partial F_1 \cup \partial F_2) = -\Delta n_1 n_2$. Therefore

$$\begin{aligned}\chi(X) &= \chi(T) + \chi(F_1 \cup F_2) - \chi\big(T \cap (F_1 \cup F_2)\big) \\ &= \frac{\Delta n_1 n_2}{2} - n_1 - n_2 \ . \quad \square\end{aligned}$$

Thus for $P(6)$, $\chi(X) = 8$; for $P(7)$, $\chi(X) = 4$; and for $P(8)_1$ and $P(8)_2$, $\chi(X) = 12$.

We count the components of $\partial_0 N(X)$ as follows. The arcs $F_1 \cap F_2$ decompose the surfaces F_1, F_2 into faces. Each of F_1 and F_2 is locally 2-sided in $N(X)$, so $\partial_0 N(X)$ contains two copies f^+, f^- of each such face f. The number of components of the union of these $f^\pm$'s is equal to the number of components of $\partial_0 N(X)$ (which is obtained from this union by adding the 2-cells into which T is divided by $\partial F_1 \cup \partial F_2$).

Lemma 11.4. *For $P(6)$, $P(7)$, $P(8)_1$, and $P(8)_2$, the number of components of $\partial_0 N(X)$ is equal to $\chi(X)$.*

Proof. We do $P(8)_1$ and $P(7)$ as examples, leaving the other similar verifications to the reader. The case of $P(7)$ is a little different from the others inasmuch as here one of the surfaces (F_2) does not separate $N(X)$.

$\underline{P(8)_1}$. Shade the faces in F_1 and F_2 alternately black and white as shown in Figure 78. (For convenience, we shall think of the surface shown in Figure 78(i) as F_1, and that shown in Figure 78(ii) as F_2. This differs from the notation of the last paragraph of Section 6.) Then the components of $\partial_0 N(X)$ fall into four

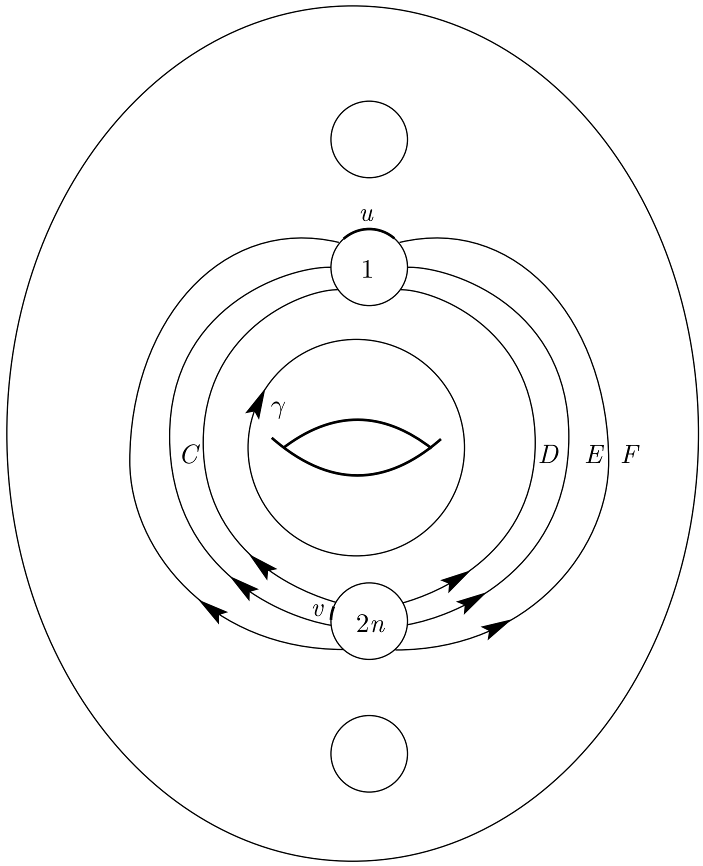


FIGURE 77

classes, defined in the obvious way: black/black, black/white, white/black, and white/white. The components in each of these classes are the unions of the following sets of faces, where we use subscripts to indicate which surface the given face lies in:

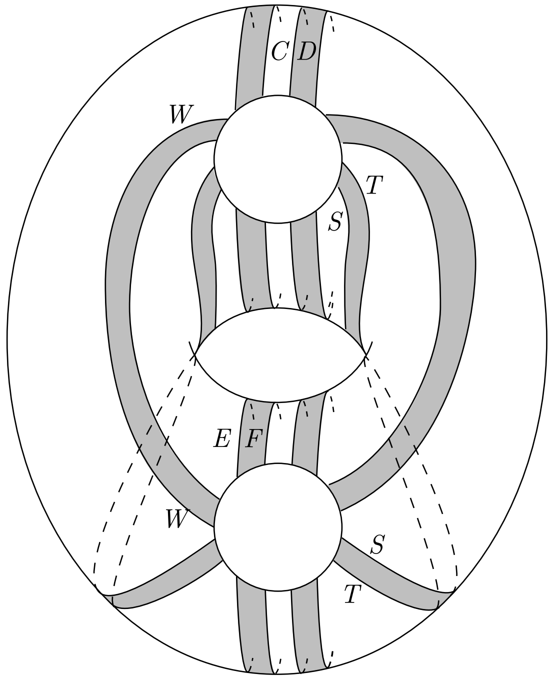
black/black: AB_1 , BG_2 , GH_1 , HA_2 ;
CD_1 , DE_2 , EF_1 , FC_2 ;
WX_1 , XV_2 , VU_1 , UW_2 ;
YZ_1 , ZT_2 , TS_1 , SY_2 .

black/white: AB_1 , BET_2 , EF_1 , TS_1 , FAS_2 ;
CD_1 , DWG_2 , WX_1 , GH_1 , XHC_2 ;
UV_1 , VZ_2 , ZY_1 , YU_2 .

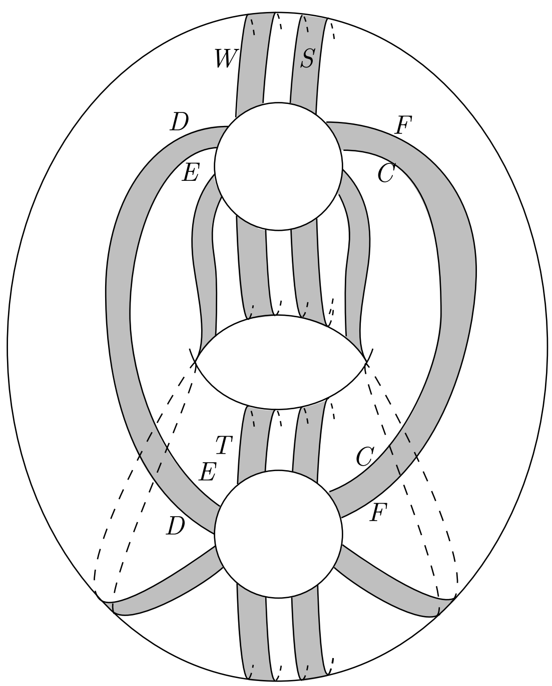
white/black: WU_2 , UTH_1 , TZ_2 , HA_2 , ZWA_1 ;
YS_2 , SVD_1 , VX_2 , DE_2 , XYE_1 ;
CF_2 , FG_1 , GB_2 , BC_1 .

white/white: BC_1 , CHX_2 , HUT_1 , XEY_1 , UY_2 , TBE_2 ;
FG_1 , GDW_2 , DSV_1 , WZA_1 , SAF_2 , VZ_2 .

This gives a total of 12, as claimed.



(i)



(ii)

Figure 78

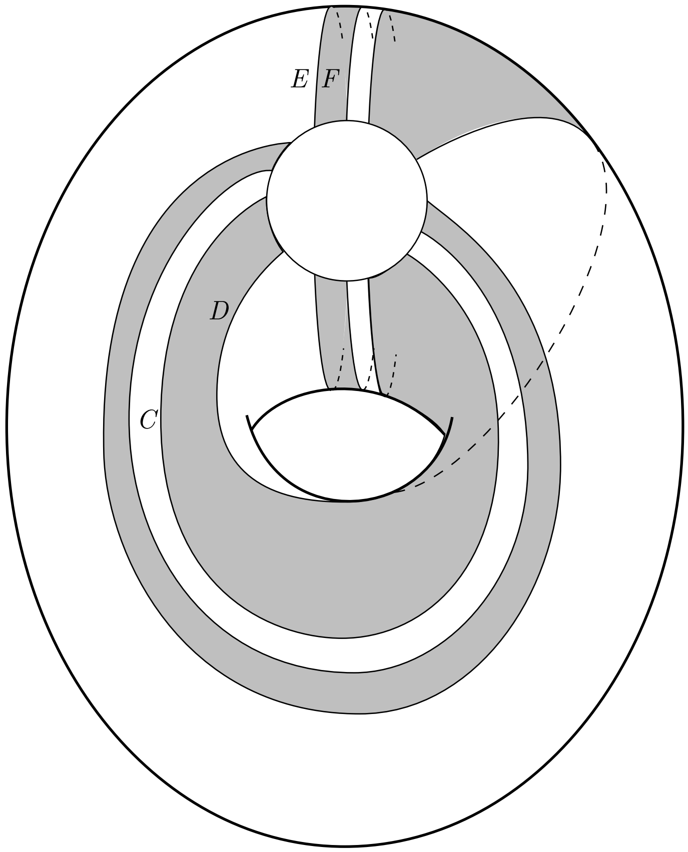


FIGURE 79

P(7). Since F_1 does separate $N(X)$, we can shade the faces on F_2 alternately black and white as shown in Figure 79. Although F_2 does not separate $N(X)$, it is of course locally 2-sided. We designate the two sides $+$ and $-$, and if f is a face on F_2 then we denote by $f^{\pm}$ the push-off of f in the appropriate direction. Then (referring also to Figure 35(i)) the components of $\partial_0 N(X)$ are the unions of the following sets of faces, where we first list those involving black faces of F_2, and second those involving white faces of F_2:

black: AB_2^+ , BF_1 , FE_2^- , EA_1 ;
CDG_2^+ , $GDEB_1$, CG_1 , DGC_2^- , EF_2^+ , BA_2^- , $FCDA_1$.

white: BC_2^+ , CG_1 , GF_2^- , FB_1 ;
DEA_2^+ , $EBGD_1$, AE_1 , EAD_2^- , BC_2^- , FG_2^+ , $ADCF_1$.

This gives a total of four components, as claimed. □

Proof of Proposition 1.5. Suppose that (M,T) is not cabled, and let $(F_\alpha, \partial F_\alpha) \subset (M,T)$ be an essential punctured torus with boundary slope r_α, $\alpha = 1,2$, such that $\Delta(r_1, r_2) \geq 6$. After the work in the previous sections we know that $(F_1, F_2; F_1 \cap F_2)$ has to be given by one of the four identification patterns $P(6)$, $P(7)$, $P(8)_1$, or $P(8)_2$. Let N be a regular neighborhood of $F_1 \cup F_2 \cup T$ in M. Then N is homeomorphic to the appropriate $N(X)$ described above. By Lemma 11.2, $\partial N - T$ consists of 2-spheres. Since M is irreducible, $M = N \cup$ 3-balls. Hence $(M; F_1, F_2)$ is uniquely determined, up to homeomorphism, in each of the four cases. It follows that these must correspond to the examples described in Section 1, with $M = W(2)$, $W(-5/2)$, $W(1)$, and $W(-5)$. □

Remark. The identification patterns $P(8)_1$ and $P(8)_2$ are distinguishable by the fact that for $P(8)_1$, $d = 1$, whilst for $P(8)_2$, $d = 3$. Also, with suitable parametrizations, the slopes $r, s \in \mathcal{T}(M, \partial M)$ with $\Delta(r, s) = 8$ are $4, -4$ for $M = W(1)$, and $\frac{1}{2}, -\frac{3}{2}$ for $M = W(-5)$. Hence the punctured tori in $W(1)$ correspond to $P(8)_1$, and those in $W(-5)$, to $P(8)_2$.

12. Other surfaces with non-negative euler characteristic

We have now proved Proposition 1.5. Before discussing the case when (M, T) is cabled, we consider other surfaces of non-negative euler characteristic, for applications to Theorems 1.2, 1.3, and 1.4. In this section we shall prove certain analogs of Proposition 1.5 in these cases.

Let $\mathcal{S}(M, T)$, $\mathcal{D}(M, T)$, $\mathcal{A}(M, T)$ denote the set of boundary slopes on T of essential surface F in M, with $F \cap T \neq \emptyset$, such that $\widehat{F}$ is homeomorphic to S^2, D^2, or the annulus A^2, respectively.

Proposition 12.1. *If $r \in \mathcal{S}(M, T)$ and $s \in \mathcal{T}(M, T)$ then either $\Delta(r, s) \leq 5$ or (M, T) is cabled.*

Proof. This follows from [GLi, Proposition 6.1]. □

For M hyperbolic with $\partial M = T$, this has recently been improved to $\Delta(r, s) \leq 4$ by Boyer and Zhang [BZ2], and subsequently to $\Delta(r, s) \leq 3$ by Oh [O] and by Wu [Wu2].

Proposition 12.2. *If $r \in \mathcal{S}(M, T) \cup \mathcal{T}(M, T)$ and $s \in \mathcal{D}(M, T) \cup \mathcal{A}(M, T)$, then either $\Delta(r, s) \leq 5$ or (M, T) is cabled.*

Proof. Let F_α be an essential surface in M, such that $F_\alpha \cap T \neq \emptyset$, with boundary slope r_α on T, $\alpha = 1, 2$, and such that $\widehat{F}_1 \cong S^2$ or T^2 and $\widehat{F}_2 \cong D^2$ or A^2. By considering the arcs of $F_1 \cap F_2$ we get graphs $\Gamma_\alpha \subset \widehat{F}_\alpha$, $\alpha = 1, 2$. If $\widehat{F}_1 \cong S^2$, then we can formally add a handle to $\widehat{F}_1$ and regard Γ_1 as a graph in T^2. Similarly, if $\widehat{F}_2 \cong D^2$, we can remove a small open disk from $\widehat{F}_2$ and regard Γ_2 as a graph in A^2. So without loss of generality we assume that $\Gamma_1 \subset T^2$ and $\Gamma_2 \subset A^2$.

Suppose that $\Delta = \Delta(r_1, r_2) \geq 6$. By Lemmas 3.1 and 3.2, either Γ_1 has $n_2 + 1$ parallel edges, or $\Delta = 6$ and each parallelism class of edges in Γ_1 has exactly n_2 members.

In the first case, adopting the notation used in the proof of Lemma 4.2, either $C_\theta(\boldsymbol{A})$ or $C_\theta(\boldsymbol{B})$ bounds a disk in the annulus $\widehat{F}_2$, or A_1 and B_1 are parallel on F_2. By Lemmas 2.1 and 2.3, we conclude that (M, T) is cabled.

In the second case, by identifying the two boundary components of $\widehat{F}_2 \cong A^2$, we can regard both Γ_1 and Γ_2 as graphs in T^2. Now the arguments of Sections 4–11 show that, unless (M, T) is cabled, the only possibility for the pair Γ_1, Γ_2 is given by the pattern $P(6)$ illustrated in Figure 28. But neither graph there is contained in an annulus. □

Proposition 12.3. *If $r, s \in \mathcal{D}(M, T) \cup \mathcal{A}(M, T)$, then either*

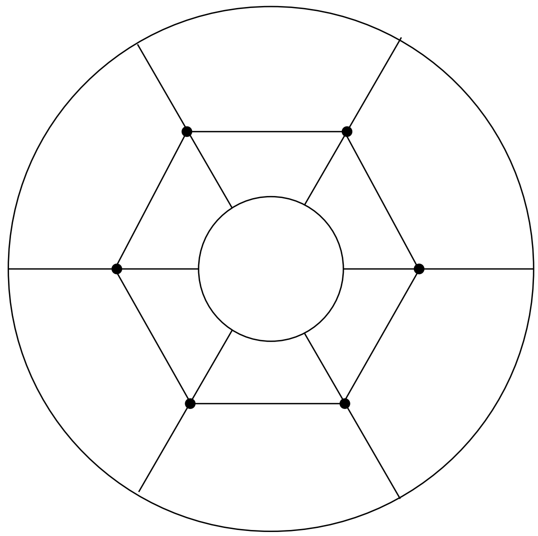


FIGURE 80

(1) $\Delta(r, s) \leq 5$; *or*
(2) (M, T) *is cabled; or*
(3) M *contains an essential annulus having exactly one boundary component on* T*, with slope* r_0 *where* $\Delta(r_0, r) = \Delta(r_0, s) = 1$; *or*
(4) M *is homeomorphic to* $T \times I$.

Proof. Let F_1, F_2 be essential surfaces in M as above such that $\widehat{F}_\alpha \cong D^2$ or A^2, $\alpha = 1, 2$. The arcs of $F_1 \cap F_2$ with at least one endpoint on T give graphs Γ_1, Γ_2 in $\widehat{F}_1, \widehat{F}_2$ respectively. As in the proof of Proposition 12.2, we may assume that $\widehat{F}_1$ and $\widehat{F}_2$ are annuli.

Suppose that $\Delta = \Delta(r_1, r_2) \geq 6$. Then by Lemma 3.3, Γ_1 has either $n_2 + 1$ parallel internal edges or $2n_2$ parallel boundary edges.

In the first case we conclude that (M, T) is cabled exactly as in the proof of Proposition 12.2.

In the second case, let $A_1, \ldots, A_{n_2}$, $B_1, \ldots, B_{n_2}$ be the corresponding parallel arcs of $F_1 \cap F_2$ on F_1, numbered in order so that (without loss of generality) A_i and B_i each has label i at its endpoint on $\partial F_1 \cap T$, $1 \leq i \leq n_2$. If, for some i, both A_i and B_i go to the same boundary component of the annulus $\widehat{F}_2$, then we get either conclusion (3) or conclusion (4), as in [CGLS, Lemmas 2.5.4 and 2.5.5]. So we may suppose that in Γ_2, from each vertex there are boundary edges going to both boundary components of $\widehat{F}_2$. Thus the reduced graph $\overline{\Gamma}_2$ is a subgraph of a graph of the form illustrated in Figure 80.

If Γ_2 has $n_1 + 1$ parallel internal edges, then we conclude as before that (M, T) is cabled. Also, if Γ_2 has $2n_1 + 1$ parallel boundary edges, then two of these must correspond to edges in Γ_1 joining some vertex to the same boundary component of $\widehat{F}_1$, so again we get conclusion (3) or (4) as in [CGLS, Lemmas 2.5.4 and 2.5.5]. It follows that $\overline{\Gamma}_2$ is as illustrated in Figure 80, $\Delta = 6$, each parallelism class of internal edges of Γ_2 has exactly n_1 members, and each parallelism class of boundary edges

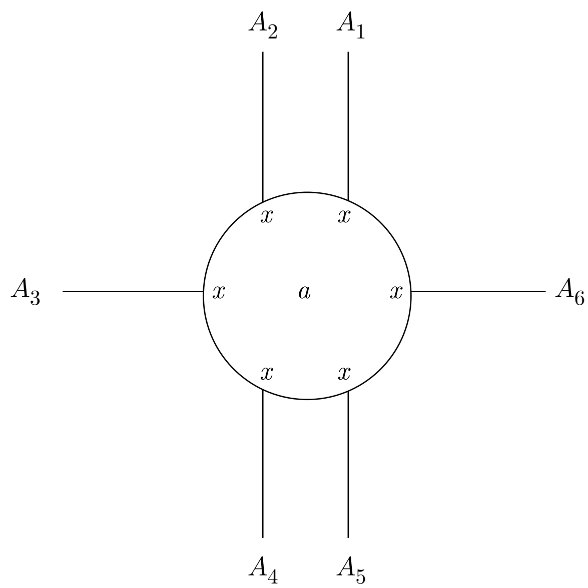


FIGURE 81

has exactly $2n_1$ members. Since the argument is symmetrical, the corresponding statement also holds for Γ_1.

Now let a be a component of $\partial F_1 \cap T$, and let $A_1, A_2, \ldots, A_6$ be the arcs of $F_1 \cap F_2$ with label x at a, numbered cyclically around a so that A_1, A_2, A_4, and A_5 are boundary edges and A_3, A_6 are internal edges (see Figure 81). On F_2, A_1 and A_2 must go to distinct boundary components of $\widehat{F}_2$ (otherwise we get conclusion (3) or (4) by [CGLS, Lemma 2.5.4]), and similarly for A_4 and A_5. But since $d = 1$, $A_1, A_2, \ldots, A_6$ occur in the same cyclic order around x in F_2. This is clearly a contradiction. □

13. CABLED MANIFOLDS

Let (M, T) be cabled, so $M = M' \cup C$, where C is a (p, q)-cable space, $\partial C = T \amalg T'$, and $M' \cap C = T'$. Let F be an essential surface in M, isotoped so as to minimize the number of components of $F \cap T'$. Then $F \cap C$ and $F' = F \cap M'$ are essential in C and M' respectively. Now suppose that $F \cap T \neq \emptyset$ and that $F \cap C$ is planar. The essential planar surfaces in a cable space are described in [GLi, Lemma 3.1]. In particular, adopting the terminology of that lemma, there are four possibilities for $F \cap C$:

an annulus of type (1);
a number of annuli of type (3), possibly together with some annuli of type (2);
a number of parallel copies of a surface of type (4);
a number of parallel copies of a surface of type (5).

We shall say that F is of type (1), (3), (4), or (5) respectively.

Throughout this section, we refer the reader to [GLi, Lemma 3.1] for the descriptions of the boundary slopes of planar surfaces in cable spaces.

Lemma 13.1. *Let r_α, r'_α be the inner and outer boundary slopes of a planar surface of type (4) in a (p,q)-cable space, $\alpha = 1,2$, with $r_1 \neq r_2$.*

(i) *There exists a slope r_0 on T such that $\Delta(r_0, r_\alpha) = 1$, $\alpha = 1,2$.*
(ii) *There exists a slope r'_0 on T' such that $\Delta(r'_0, r'_\alpha) = 1$, $\alpha = 1,2$, if and only if $q = 2$ and $\Delta(r_1, r_2) = 1$.*

Proof. (i) With respect to the standard cable space coordinates,

$$r_\alpha = (1 + k_\alpha pq)/k_\alpha,$$

$\alpha = 1,2$, for some integers k_1, k_2. Then $r_0 = pq$ (the slope of an ordinary fibre in the Seifert fibre space structure) satisfies $\Delta(r_0, r_\alpha) = 1$, $\alpha = 1,2$.

(ii) With respect to the standard cable space coordinates, $r'_\alpha = (1 + k_\alpha pq)/k_\alpha q^2$. Suppose $r'_0 = x/y$ satisfies $\Delta(r'_0, r'_\alpha) = 1$, $\alpha = 1,2$. Then

$$(1 + k_\alpha pq)y - k_\alpha q^2 x = \pm 1 \ ,$$

that is

$$y + k_\alpha q(py - qx) = \pm 1 \ , \qquad \alpha = 1,2 \ .$$

Subtracting, we obtain

$$(k_1 - k_2)q(py - qx) = 0 \ \text{ or } \ \pm 2 \ .$$

The first case gives $py - qx = 0$, and (hence) $y = \pm 1$, contradicting the fact that $(p,q) = 1$. The second case gives $q = 2$ and $|k_1 - k_2| = \Delta(r_1, r_2) = 1$. Conversely, if $q = 2$ and $|k_1 - k_2| = 1$ then one easily verifies that an r'_0 with the desired property exists. □

Let F_1, F_2 be essential surfaces in M as above with boundary slopes r_1, r_2 on T, and let $\Delta = \Delta(r_1, r_2)$. The following lemma is an immediate consequence of the description of the boundary slopes in [GLi, Lemma 3.1].

Lemma 13.2. *Let M, F_1, F_2 be as above.*

(a) *If F_α is of type (1) or (3), $\alpha = 1,2$, then $r_1 = r_2$.*
(b) *If F_1 is of type (1) or (3) and F_2 is of type (4), then $\Delta = 1$.*
(c) *If F_1 is of type (1) or (3) and F_2 is of type (5), then $\Delta = q$.*
(d) *If F_1 is of type (4) and F_2 is of type (5), then $(\Delta, q) = 1$.*

In the next two lemmas, F_1 and F_2 are essential surfaces in $M = M' \cup C$, with boundary slopes r_1, r_2 on T and r'_1, r'_2 on T'. We assume $r_1 \neq r_2$ and write $\Delta = \Delta(r_1, r_2)$, $\Delta' = \Delta(r'_1, r'_2)$. We also assume that (M', T') is cabled, the corresponding cable space being C'.

The calculation in the proof of Lemma 13.1(ii) is very similar to that given in [GLi, p.137], where it is used to prove the next lemma.

Lemma 13.3. *Let M, F_1, F_2 be as above. If (M', T') is cabled and F_1, F_2, F'_1, F'_2 are of type (4), then $q = 2$ and $\Delta = 1$.*

Proof. This follows from Lemma 13.1. □

Lemma 13.4. *Let M, F_1, F_2 be as above, where F_1 and F_2 are of type (4). Suppose that (M', T') is $(1,2)$-cabled.*

(a) *If F'_1 is of type (4) and F'_2 is of type (5), then $\Delta = 1$.*

(b) *If F_1' and F_2' are of type (5), then either $q = 2$ and the identification of C and C' along T' preserves the fibres of the Seifert fibrations of C and C', or $\Delta \le 2$.*

Proof. With respect to the standard cable coordinates on C, $r_\alpha' = (1 + k_\alpha pq)/k_\alpha q^2$, $\alpha = 1, 2$.

The identification of the outer boundary of C with the inner boundary of C' is given by a matrix $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in GL(2, \mathbb{Z})$.

(a) With respect to the standard cable coordinates on C',

$$r_1' = (1 + 2k)/k \ ,$$
$$r_2' = (2 + 2m)/m \ ,$$

for some integers k and m.

Applying the identification matrix to the C-coordinates of r_1' and r_2' gives the four equations

$$\text{(i)} \qquad x(1 + k_1 pq) + yk_1 q^2 = \varepsilon_1(1 + 2k),$$
$$\text{(ii)} \qquad z(1 + k_1 pq) + wk_1 q^2 = \varepsilon_1 k,$$
$$\text{(iii)} \qquad x(1 + k_2 pq) + yk_2 q^2 = \varepsilon_2(2 + 2m),$$
$$\text{(iv)} \qquad z(1 + k_2 pq) + wk_2 q^2 = \varepsilon_2 m,$$

where $\varepsilon_\alpha = \pm 1$, $\alpha = 1, 2$. Taking ((i)–2(ii))–((iii)–2(iv)), we get

$$(k_1 - k_2)q\big(p(x - 2z) + q(y - 2w)\big) = \varepsilon_1 - 2\varepsilon_2 = -1 \quad \text{or} \quad \pm 3 \ .$$

Therefore $|k_1 - k_2| = \Delta = 1$ (and $q = 3$).

(b) With respect to the standard cable coordinates on C',

$$r_\alpha' = (2 + 2m_\alpha)/m_\alpha \ , \qquad \alpha = 1, 2 \ ,$$

for some integer m_1, m_2.

Hence, as in (a) above, we get two equations

$$\text{(i)}_\alpha \qquad x(1 + k_\alpha pq) + yk_\alpha q^2 = \varepsilon_\alpha(2 + 2m_\alpha),$$
$$\text{(ii)}_\alpha \qquad z(1 + k_\alpha pq) + wk_\alpha q^2 = \varepsilon_\alpha m_\alpha,$$

for each $\alpha = 1, 2$. Subtracting (i)$_2$ from (i)$_1$ and (ii)$_2$ from (ii)$_1$ gives

$$\text{(v)} \qquad (k_1 - k_2)q(xp + yq) = 2\big((\varepsilon_1 - \varepsilon_2) + (\varepsilon_1 m_1 - \varepsilon_2 m_2)\big),$$
$$\text{(vi)} \qquad (k_1 - k_2)q(zp + wq) = \varepsilon_1 m_1 - \varepsilon_2 m_2.$$

If $\varepsilon_1 = \varepsilon_2$, then these equations show that

$$xp + yq = 2(zp + wq) \ .$$

Since $(xp + yq, zp + wq) = 1$, this implies that

$$zp + wq = \varepsilon \ , \quad xp + yq = 2\varepsilon \ , \quad \text{where} \quad \varepsilon = \pm 1 \ ;$$

in other words,

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 2\varepsilon \\ \varepsilon \end{bmatrix} \ .$$

Since the slopes on T' of the fibres of C and C' are p/q and $2/1$ respectively, this shows that the identification of the outer boundary of C with the inner boundary of C' is fibre-preserving. Also, equation (vi) now gives

$$|k_1 - k_2|q = |m_1 - m_2| \ .$$

But $|k_1 - k_2|q^2 = \Delta' = 2|m_1 - m_2|$. Therefore $q = 2$.

If $\varepsilon_1 = -\varepsilon_2$, then (v)-2(vi) gives an equation of the form

$$(k_1 - k_2)qX = \pm 4 \ .$$

Hence $\Delta = |k_1 - k_2| \leq 2$. □

Lemma 13.5. *Let M be a Seifert fibre space with orbit surface a disk and two exceptional fibres of multiplicities q_1, q_2. Then M contains exactly two essential surfaces, one a vertical annulus and the other horizontal, with boundary slopes r_0 and r_1, say. Moreover if $r_1 \in \mathcal{S}(M, \partial M)$ then $\Delta(r_0, r_1) = 1$, and if $r_1 \in \mathcal{T}(M, \partial M)$ then $\{q_1, q_2, \Delta(r_0, r_1)\} = \{3,3,3\}$, $\{2,4,4\}$, or $\{2,3,6\}$.*

If $q_1 = 2$ and $q_2 = 3$ then M is homeomorphic to the exterior of the trefoil knot, so the following corollary is immediate. It will be used in the proof of Proposition 13.7.

Corollary 13.6. *Let M be as in Lemma 13.5, and suppose $r, s \in \mathcal{S}(M, \partial M) \cup \mathcal{T}(M, \partial M)$. Then either $\Delta(r, s) \leq 4$, or M is homeomorphic to the exterior of the trefoil knot, r (say) $\in \mathcal{S}(M, \partial M)$, $s \in \mathcal{T}(M, \partial M)$, and $\Delta(r, s) = 6$.*

Proof of Lemma 13.5. Note that $M = M' \cup C$, where C is a (p_1, q_1)-cable space and M' is a solid torus. Let F be an essential surface in M, isotoped so as to intersect M' minimally. Then $F \cap C$ is an essential surface in C, and is therefore either a vertical annulus or horizontal. In the second case the boundary slope of $F \cap C$ on $\partial M'$ must be that of a meridian disk of M'. It then follows from [GLi, proof of Lemma 3.1] that $F \cap C$, and hence F, is uniquely determined.

If $r \in \mathcal{S}(M, \partial M)$, then $\Delta(r_0, r) = 1$ by [GLi, Proposition 1.4].

If $r \in \mathcal{T}(M, \partial M)$, and F is the punctured torus in M with boundary slope r, then $\widehat{F}$ is a horizontal torus in $M(r)$. Note that $M(r)$ is a Seifert fibre space with orbit surface S^2 and (at most) 3 exceptional fibres of multiplicities q_1, q_2, and $q_3 = \Delta(r_0, r) \geq 1$. Hence $\widehat{F}$ is a k-fold covering of S^2 branched over 3 points x_1, x_2, x_3 with branching indices q_1, q_2, q_3. Let b_i be the number of lifts of x_i in $\widehat{F}$, $i = 1, 2, 3$. Then $k = b_i q_i$, $i = 1, 2, 3$. Also,

$$\begin{aligned} 0 = \chi(\widehat{F}) &= k\chi\big(S^2 - \{x_1, x_2, x_3\}\big) + b_1 + b_2 + b_3 \\ &= -k + b_1 + b_2 + b_3 \\ &= k\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} - 1\right) \ . \end{aligned}$$

Therefore $\dfrac{1}{q_1} + \dfrac{1}{q_2} + \dfrac{1}{q_3} = 1$, giving the three euclidean triples listed. □

Let $\mathcal{B}_+(M, T)$ denote the set of boundary slopes on T of essential surfaces F in M with $\chi(\widehat{F}) \geq 0$, that is, $\mathcal{B}_+(M, T) = \mathcal{S}(M, T) \cup \mathcal{T}(M, T) \cup \mathcal{D}(M, T) \cup \mathcal{A}(M, T)$.

Proposition 13.7. *Suppose that (M, T) is cabled and that ∂M is incompressible. If $r, s \in \mathcal{B}_+(M, T)$, then either*

(1) $\Delta(r, s) \leq 4$; *or*

(2) *M is homeomorphic to the exterior of the trefoil knot, r (say) $\in \mathbb{S}(M,T)$, $s \in \mathcal{T}(M,T)$, and $\Delta(r,s) = 6$; or*
(3) *M is a cable space, and $r, s \in \mathcal{D}(M,T) \cup \mathcal{A}(M,T)$; or*
(4) *M is a Seifert fibre space with orbit surface an annulus and two exceptional fibres of multiplicity 2, and $r, s \in \mathcal{A}(M,T)$.*

Before proving this proposition, we observe the following corollary, in which we drop the assumption that ∂M is incompressible. Note that Proposition 1.6 is an immediate consequence.

Corollary 13.8. *If (M,T) is cabled and $r, s \in \mathcal{B}_+(M,T)$ then either conclusion (1) or (2) of Proposition 13.7 holds, or $r, s \in \mathcal{D}(M,T) \cup \mathcal{A}(M,T)$ and M contains an essential annulus with exactly one boundary component on T.*

Proof. If ∂M is incompressible, then the corollary follows immediately from Proposition 13.7.

If ∂M is compressible, let W be a maximal compression body for $\partial M - T$ in M, and let $M_0 = \overline{M - W}$. If F is an essential surface in M (such that $\partial F \cap T \neq \emptyset$), isotoped so as to minimize the number of components of $F \cap \partial M_0$, then $F \cap M_0$ and $F \cap W$ are essential. Hence each component of $F \cap W$ is an annulus with one boundary component on ∂M_0 and the other on ∂M. If $r, s \in \mathcal{B}_+(M,T)$ we can therefore apply Proposition 13.7 to the corresponding surfaces $F \cap M_0$ in M_0. If conclusion (1) or (2) holds for M_0, then it holds for M. If conclusion (3) or (4) holds for M_0, then there is an essential annulus in M_0 with one boundary component on T and the other, γ, say, on $\partial M_0 - T$. Since there is an annulus in W with one boundary on ∂M and the other equal to γ, we get an annulus in M as described. □

Proof of Proposition 13.7. Let F_1, F_2 be essential surfaces in $M = M' \cup C$ as above, such that $\widehat{F}_\alpha \cong S^2, T^2, D^2$, or A^2, $\alpha = 1, 2$. Let r_α be the boundary slope of F_α on T, and r'_α the boundary slope of $F'_\alpha = F_\alpha \cap M'$ on T' (if $F'_\alpha \neq \emptyset$), $\alpha = 1, 2$. Let $\Delta = \Delta(r_1, r_2)$ and $\Delta' = \Delta(r'_1, r'_2)$. We suppose $r_1 \neq r_2$, therefore $r'_1 \neq r'_2$ (see [GLi, proof of Lemma 3.1]).

If $\widehat{F}_1 \cong \widehat{F}_2 \cong S^2$, then $\Delta = 1$ by [GLi, Proposition 1.4], so we need not consider this case.

First note that if some component of $F_\alpha \cap C$ is a punctured torus, then some component of F'_α is a disk, and hence M' is a solid torus. Therefore M is a Seifert fibre space over the disk with two exceptional fibres, and the result follows from Corollary 13.6. We may therefore assume that all components of $F_1 \cap C$ and $F_2 \cap C$ are planar.

If F_α is of type (5), then, by considering the euler characteristics of F_α, $F_\alpha \cap C$, and F'_α, we easily conclude the following:

(i) the case $\widehat{F}_\alpha \cong D^2$ is impossible;
(ii) if $\widehat{F}_\alpha \cong S^2$, then F_α is a disk, contradicting the incompressibility of T;
(iii) if $\widehat{F}_\alpha \cong T^2$ or A^2, then $q = 2$ and F'_α consists of annuli.

In case (iii), suppose that $\widehat{F}_\alpha \cong A^2$. Then some component E_α, say, of F'_α has exactly one boundary component on T'. If $\widehat{F}_\beta \cong T^2$ and $F'_\beta \neq \emptyset$, then by considering $E_\alpha \cap F'_\beta$ we see that F'_β is boundary-compressible, and hence compressible, a contradiction. If $\widehat{F}_\beta \cong A^2$, then F'_β has a component E_β with exactly one boundary component on T'. By hypothesis, the annuli E_1 and E_2 have distinct boundary slopes on T', and hence it easily follows, using the irreducibility of M and the incompressibility of ∂M, that $M' \cong T' \times I$ (compare [CGLS, Lemma 2.5.3]). Thus M is a cable space. It now readily follows that we must have either conclusion (3) or conclusion (1).

After Lemma 13.2(a), (b), and (c), then, we may assume that we are in one of the following three cases, and that if F_α is of type (5) then $\widehat{F}_\alpha \cong T^2$:

I. F_1 and F_2 of type (4);
II. F_α of type (4) and F_β of type (5);
III. F_1 and F_2 of type (5).

First we dispose of Case III. Here $q = 2$, F_1' and F_2' consist of annuli with their boundaries on T', and $\Delta = 4\Delta'$. By [GLi, p.139], $\Delta' = 1$, and hence $\Delta = 4$.

To treat Cases I and II, we proceed by induction on the *cable length* of (M, T), which is defined in the obvious way as follows: if (M, T) is not cabled it has cable length 0, and if (M, T) is a cabling of (M', T') then cable length (M, T) = cable length $(M', T') + 1$.

Let us consider Case II. Here $q = 2$, F_β' consists of annuli with their boundaries on T', and $\Delta' = \Delta$. Let E be a component of F_β' and consider $E \cap F_\alpha'$. If $2\Delta' > 6$ then some pair of arcs of intersection must be parallel on F_α'. (If $\widehat{F}_\alpha' \cong T^2$, this follows from Lemma 3.1. The case $\widehat{F}_\alpha' \cong S^2$ also follows formally from that lemma by adding a handle. Similarly, the cases $\widehat{F}_\alpha' \cong A^2$ or D^2 follow by embedding A^2 or D^2 in T^2, where, in the case of A^2, the embedding is chosen to be essential.) Since all arcs of intersection are necessarily parallel on E, we conclude from Lemma 2.1 that either $\Delta = \Delta' \leq 3$ or (M', T') is cabled. In the second case we may assume by induction, since $r_\beta' \in \mathcal{S}(M', T')$, that either $\Delta' \leq 4$ or $\Delta' = 6$. But by Lemma 13.2(d), Δ is odd. Hence $\Delta \leq 3$.

Finally we consider Case I. Here $\Delta' = q^2\Delta$. By Propositions 1.5, 12.1, 12.2, and 12.3, either

(i) $\Delta' \leq 8$; or
(ii) (M', T') is cabled; or
(iii) there exists a slope r_0' on T' such that $\Delta(r_0', r_\alpha') = 1$, $\alpha = 1, 2$; or
(iv) M' is homeomorphic to $T' \times I$.

In case (i), we get $\Delta \leq 2$.

In case (iii), we get $\Delta = 1$ by Lemma 13.1(ii).

In case (iv), M is a cable space and we get conclusion (3).

In case (ii), we assume by induction that Proposition 13.7 holds for (M', T'). Thus either $\Delta' \leq 6$ (in which case $\Delta = 1$), or conclusion (3) or (4) holds for (M', T').

First suppose that conclusion (3) holds. Then $M = C \cup C'$, where C' is a (p', q')-cable space, say, and $r_1', r_2' \in \mathcal{D}(C', T') \cup \mathcal{A}(C', T')$. We consider the various possibilities for F_1' and F_2', which must be of type (3), (4), or (5). Recall that if F_α' is of type (5) then $q' = 2$.

If F_α' is of type (3), then $\Delta' \leq 2$ by Lemma 13.2 (b) and (c), a contradiction.

If F_1' and F_2' are of type (4), then $\Delta = 1$ by Lemma 13.3.

If F_α' is of type (4) and F_β' is of type (5), then $\Delta = 1$ by Lemma 13.4(a).

If F_1' and F_2' are of type (5), then, by Lemma 13.4(b), either $\Delta \leq 2$ or conclusion (4) of Proposition 13.7 holds.

Finally, if conclusion (4) holds for (M', T'), then note that (by induction) we may assume that this case only arises when F_1' and F_2' are both of type (4). But then $\Delta = 1$ by Lemma 13.3. □

An example where (M, T) is cabled and has $r, s \in \mathcal{T}(M, T)$ with $\Delta(r, s) = 4$ is the following. Let M' be the Seifert fibre space over the disk with two exceptional fibres of multiplicity 2. Equivalently, M' is the exterior of the (1,2)-cable of a core of $S^1 \times S^2$. Then M' contains two essential annuli (one vertical, the other horizontal) with boundary slopes r_1', r_2' such that $\Delta(r_1', r_2') = 1$. (In fact, M' is the only example of a manifold that contains two essential annuli with distinct boundary slopes; see [GLi, pp.138–139].) Now if we let $(M, \partial M)$ be the (1,2)-cabling of $(M', \partial M')$, it is easy to construct essential punctured tori F_1, F_2 in M, of type (5), with boundary slopes r_1, r_2 such that $\Delta(r_1, r_2) = 4$.

14. Final proofs

First we combine the results of Sections 12 and 13 on $\mathcal{B}_+(M, T)$.

For spheres, we recall the following, which is proved in [GLu], and also [BZ1].

Theorem 14.1. *If $r, s \in \mathcal{S}(M, T)$ then $\Delta(r, s) \leq 1$.*

Actually, the bound $\Delta(r, s) \leq 4$, obtained in [GLi, Theorem 1.1], would suffice for our present purposes.

For tori, we have Theorem 1.1, which follows from Propositions 1.5 and 1.6. The former was proved in Sections 4–11, whilst the latter is a consequence of Corollary 13.8.

The next three theorems follow from Propositions 12.1, 12.2, and 12.3 respectively, together with Corollary 13.8.

Theorem 14.2. *If $r \in \mathcal{S}(M, T)$ and $s \in \mathcal{T}(M, T)$, then either $\Delta(r, s) \leq 5$, or $\Delta(r, s) = 6$ and M is homeomorphic to the exterior of the trefoil knot.*

Theorem 14.3. *If $r \in \mathcal{S}(M, T) \cup \mathcal{T}(M, T)$ and $s \in \mathcal{D}(M, T) \cup \mathcal{A}(M, T)$, then $\Delta(r, s) \leq 5$.*

Theorem 14.4. *If $r, s \in \mathcal{D}(M, T) \cup \mathcal{A}(M, T)$ then either $\Delta(r, s) \leq 5$ or M contains an essential annulus with exactly one boundary component on T.*

In fact, if $r, s \in \mathcal{D}(M, T)$, then the bound of 5 in Theorem 14.4 can be replaced by 1 [Wu1].

It is clear that other improvements can be made to Theorems 14.2, 14.3, and 14.4, but we shall not pursue this here. Some results on $\Delta(r, s)$ for $r, s \in \mathcal{A}(M, T)$ are given in [H], and for $r \in \mathcal{D}(M, T)$, $s \in \mathcal{A}(M, T)$ in [HM].

Note that the examples listed in Theorems 1.1 and 14.2 with $\Delta(r, s) > 5$ all have $\partial M = T$. Hence, combining Theorems 1.1, 14.1, 14.2, 14.3, and 14.4, we have the following.

Corollary 14.5. *Suppose that $\partial M \neq T$. If $r, s \in \mathcal{B}_+(M, T)$, then either $\Delta(r, s) \leq 5$, or M contains an essential annulus with exactly one boundary component on T.*

We now give the proofs of Theorems 1.2, 1.3, and 1.4 stated in the Introduction.

Proof of Theorem 1.2. Suppose that $M \in \mathcal{A}$, but $M(r) \notin \mathcal{A}$. Then $M(r)$ may contain an essential sphere, in which case it is easy to show that $r \in \mathcal{S}(M, T)$. If $M(r)$ is irreducible then it must contain an essential torus, from which it easily follows (using the fact that $M(r)$ is irreducible, however) that $r \in \mathcal{T}(M,T)$. Thus if $M(r), M(s) \notin \mathcal{A}$, we must have $r, s \in \mathbf{S}(M,T) \cup \mathcal{T}(M,T)$. The result is now a consequence of Theorems 1.1, 14.1, and 14.2. □

Proof of Theorem 1.3. Suppose that $M \in \mathcal{H}_0$, where $\partial M \neq T$, and $M(r) \notin \mathcal{H}_0$. Since $\partial M(r) \neq \emptyset$, it follows from [T] that either $M(r) \notin \mathcal{A}$ (in which case $r \in \mathbf{S}(M,T) \cup \mathcal{T}(M,T)$), or $M(r)$ is Seifert fibred.

Suppose that $M(r)$ is Seifert fibred. Then $M(r)$ is irreducible and contains either an essential disk (and is homeomorphic to $S^1 \times D^2$) or an essential annulus. Therefore $r \in \mathcal{D}(M,T) \cup \mathcal{A}(M,T)$. Also, since here $\partial M(r)$ and hence ∂M consists of tori, if M contained an essential annulus it would either contain an essential torus or be Seifert fibred, contradicting our hypothesis that $M \in \mathcal{H}_0$. Therefore $M(r)$ Seifert fibred implies that $r \in \mathcal{D}(M,T) \cup \mathcal{A}(M,T)$ and that M does not contain an essential annulus.

The result now follows directly from Corollary 14.5. □

Proof of Theorem 1.4. Let M_0 be the component of M cut along S that contains $T = \partial M$. Note that M_0 is irreducible. Consider the pair (M_0, T), and let r be a slope on T. Since $M \in \mathcal{A}$, standard arguments show that if $r \notin \mathcal{B}_+(M_0, T)$ then S is incompressible in $M(r)$ and $M(r) \in \mathcal{A}$. It then follows from [T] that $M(r)$ is either hyperbolic or Seifert fibred.

Suppose that $M(r)$ is Seifert fibred. Then S is an incompressible horizontal surface in $M(r)$, so $M(r)$ cut along S, that is, $M_0(r)$, is homeomorphic to $S \times I$. Choose a set of simple closed curves in S such that S cut along these curves is a disk, and consider the corresponding annuli in $S \times I$. By considering the intersection of the solid torus V in $M_0(r) = M_0 \cup V$ with these annuli, standard arguments show that $r \in \mathcal{A}(M_0, T)$.

Hence $r \notin \mathcal{B}_+(M_0, T)$ implies $M(r) \in \mathcal{H}$. The result now follows from Corollary 14.5. □

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